THE MAXIMUM PRINCIPLES FOR PARTIALLY OBSERVED RISK-SENSITIVE OPTIMAL CONTROLS OF MARKOV REGIME-SWITCHING JUMP-DIFFUSION SYSTEM

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Abstract. This paper studies partially observed risk-sensitive optimal control problems with correlated noises between the system and the observation. It is assumed that the state process is governed by a continuous-time Markov regime-switching jump-diffusion process and the cost functional is of an exponential-of-integral type. By virtue of a classical spike variational approach, we obtain two general maximum principles for the aforementioned problems. Moreover, under certain convexity assumptions on both the control domain and the Hamiltonian, we give a sufficient condition for the optimality. For illustration, a linear-quadratic risk-sensitive control problem is proposed and solved using the main results. As a natural deduction, a fully observed risk-sensitive maximum principle is also obtained and applied to study a risk-sensitive portfolio optimization problem. Closed-form expressions for both the optimal portfolio and the corresponding optimal cost functional are obtained.

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Partial information; Risk-sensitive control; Regime-switching; Jump-diffusion; Stochastic maximum principle

1. Introduction

Optimal control problem for Markov regime-switching model has received significant attention in recent years. See, for example, [1–6], etc. Compared to the traditional models, Markov regime-switching model performs better from the empirical point of view. The basic idea of regime-switching is to modulate the model with a continuous-time finite-state Markov chain where each state represents a regime of the system or level of economic indicator. For example, in the stock market, the up-trend volatility of a stock tends to be smaller than its down-trend volatility (see [7] for further details), therefore, it is reasonable to describe the market trends by a two-state Markov chain. The regime-switching model has also been applied in the fields of option pricing (see [8]), risk management (see [9]), Markowitz mean-variance problem (see [10]), etc.

Risk-sensitive control is a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization. Since the early work of [11], the subject of risk-sensitive controls has been discussed by many researchers. One reason for looking at risk-sensitive optimal control problems is that the theory in itself is interesting and challenging. Another reason is that the risk-sensitive parameter can describe the risk attitude of an investor, thus such model can be used to study some portfolio optimization problems. Related works include [12–16] and references therein.

There are two main approaches to solve risk sensitive optimal control problems: the dynamic programming principle and the stochastic maximum principle. Although dynamic programming principle has been the tool predominantly used to study the risk-sensitive controls, several papers have been devoted to the maximum principle. In [17], the author used a heuristic approach based on large deviation theory to derive a maximum principle, where the diffusion coefficient does not contain the control variable. Combining the logarithmic transformation with the relationship between adjoint variables and the value function, a new risk-sensitive maximum principle for

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controlled diffusions with a control dependent diffusion coefficient was obtained by [18]. Then [19] and [20] extended the results in [18] to the jump-diffusion case without and with regime-switching, respectively.

However, the above mentioned works all assume that the overall information is available to controllers, i.e., controllers can fully observe the information of noisy functions of state equations. This assumption is not always satisfied in reality. In general, controllers can only get partial information in most cases. Then it is natural to study optimal control problems under partial observation. There is a rich literature on stochastic control problems under partial information. See [21–24] and references therein. Combining Girsanov's theorem with a classical method used in the full information case, [25] derived a general maximum principle for partially observed optimal control when the noises between the system and the observation are correlated. However, there is only a few papers dealing with partially observed risk-sensitive optimal control problems. A partial information maximum principle for a class of risk-sensitive controls driven by controlled diffusion models was derived by [26]. The work [27] extended the results in [26] to a more general performance criterion case. However, it appears that the partial information risk-sensitive maximum principle for Markov regime-switching jump-diffusion processes with correlated noises between the system and the observation has not yet been done and this is the main goal of this paper.

In this paper, we consider a general case of the partially observed risk-sensitive optimal control problem, where the state is governed by a continuous-time Markov regime-switching jump-diffusion process, the control is allowed to enter into all the coefficients, the correlated noises between the system and the observation is present, and the control domain is not necessarily convex. Two general maximum principles are proved. More specifically, we first introduce the first-order and second-order variational equations by virtue of a classical spike variational approach, then we derive the corresponding adjoint equations which are finite-dimensional backward stochastic differential equations (BSDEs). This is a standard method used to deal with the risk-neutral case. Confer to [24] and [28], where some partially observed and fully observed maximum principles were obtained, respectively. In order to deal with the term produced by partial information, we also introduce an adjoint BSDE that depends on the risk-sensitive parameter. Our method does not involve Zakai equations, thus we can avoid the stochastic calculus in infinite dimensional spaces, in contrast with [23]. One significant feature of our work is that we need to introduce four adjoint BSDEs when the performance criterion is a general running cost functional. This is essentially different from the full information risk-neutral case, in which only two adjoint equations are used. See, for example, [29] or [30]. Under additional convexity assumptions on both the control domain and the Hamiltonian, we further prove that the maximum principles are also sufficient. Finally, a fully observed risk-sensitive maximum principle is obtained as a natural deduction of our main results.

The other motivation of this paper is to solve a risk-sensitive portfolio optimization problem when the stock is modeled by a Markov regime-switching diffusion process. Such problem was considered in [31] using a dynamic programming approach. In this paper, we apply the risk-sensitive stochastic maximum principle to find the optimal portfolio that maximizes the risk sensitivity of an investor in such environment. Closed-form expressions for both the optimal portfolio and the corresponding optimal cost functional are obtained and given in terms of the solution of a Markov regime-switching ordinary differential equation. We also apply our main results to solve a linear-quadratic (LQ) risk-sensitive optimal control problem.

The paper is organized as follows. In Section 2, we present the formulation of our partially observed risk-sensitive optimal control problem and the main assumptions. In Section 3, we derive two partially observed risk-sensitive maximum principles and develop a sufficient condition for the optimality under some convexity assumptions. As an example, Section 4 examines an LQ risk-sensitive optimal control problem using the maximum principle derived. Section 5 is devoted to obtain a general fully observed risk-sensitive maximum principle. The latter result is applied to solve a risk-sensitive portfolio optimization problem in Section 6.

In the rest of our paper, we shall adopt the following notations.

$M^\top$:	the transpose of any vector or matrix M;
$\mathrm{tr}(M)$:	the trace of a square matrix M;
$\langle x, y\rangle$:	the inner product of $x, y \in \mathbb{R}^L$, that is $\langle x, y\rangle := x^\top y$;
$\lvert x\rvert$:	the Euclidian norm of $x \in \mathbb{R}^L$, if $x \in \mathbb{R}^{L\times N}$, $\lvert x\rvert = \sqrt{\mathrm{tr}(xx^\top)}$;
$I_A(\cdot)$:	the indicator function of the set A.

2. Formulation of The Optimal Control Problem

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space and $T > 0$ be a finite-time horizon, where $\mathbb{F} := \{\mathcal{F}_t | t \in [0, T]\}$ is a right-continuous, P-completed filtration to which all of the processes defined below including the Markov chain, the Brownian motions and the Poisson random measures are adapted.

We consider an irreducible homogeneous, continuous-time Markov chain $\{\alpha(t)|t \in [0, T]\}$ with a finite-state space $S := \{e_1, e_2, \ldots, e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$ and the jth component of e_i is the Kronecker delta δ_{ij}, for each $i, j = 1, \ldots, D$. Here, we denote by $\mathbb{R}$ the set of real numbers and $\mathbb{N}$ the set of natural numbers. The Markov chain is characterized by its Q-matrix $\Lambda := [\lambda_{ij}]_{i,j=1,\ldots,D}$ under P. Here, for each $i, j = 1, \ldots, D$, λ_{ij} is the transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij} \geq 0$, for $i \neq j$ and $\sum_{j=1}^D \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows for each $i, j = 1, \ldots, D$ with $i \neq j$, we suppose that $\lambda_{ij} > 0$, and so $\lambda_{ii} < 0$.

It follows from [32] that the following semimartingale representation of the Markov chain $\{\alpha(t)|t \in [0, T]\}$ holds

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^\top \alpha(s)\mathrm{d}s + M(t), \tag{2.1}$$

where $\{M(t)|t \in [0, T]\}$ is an $\mathbb{R}^D$-valued, $(\mathbb{F}, P)$-martingale.

Now, let us introduce a set of Markov jump martingales associated with the chain α, which will be used to model the controlled state process. For each $i, j = 1, \ldots, D$, with $i \neq j$ and $t \in [0, T]$, let $J^{ij}(t)$ be the number of jumps from state e_i to state e_j up to time t. One can show using the results in [32] that

$$J^{ij}(t) = \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle \,\mathrm{d}s + m_{ij}(t), \tag{2.2}$$

where $m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i\rangle \langle \mathrm{d}M(s), e_j\rangle$ is an $(\mathbb{F}, P)$-martingale.

For each fixed $j = 1, \ldots, D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t. Then

$$\Phi_j(t) = \sum_{i=1, i\neq j}^{D} J^{ij}(t) := \widetilde{\Phi}_j(t) + \lambda_j(t), \tag{2.3}$$

where $\widetilde{\Phi}_j(t) := \sum_{i=1,i\neq j}^D m_{ij}(t)$ and $\lambda_j(t) := \sum_{i=1,i\neq j}^D \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle \,\mathrm{d}s$. Note that, for each $j = 1, \ldots, D$, $\widetilde{\Phi}_j(t)$ is again an $(\mathbb{F}, P)$-martingale.

In what follows, let $L, M, N, K \in \mathbb{N}$. Suppose that $N^i(\mathrm{d}t, \mathrm{d}\zeta), i = 1, \ldots, M$, are independent Poisson random measures on $(\mathbb{R}_+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0))$, with $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Here $\mathcal{B}(\mathbb{R}_+)$ and $\mathcal{B}(\mathbb{R}_0)$ are the Borel σ-fields generated by open subsets of $\mathbb{R}_+$ and $\mathbb{R}_0$ respectively. Assume that the Poisson random measure $N^i(\mathrm{d}t, \mathrm{d}\zeta)$ has the following compensator

$$n^i_\alpha(\mathrm{d}t, \mathrm{d}\zeta) := \nu^i_\alpha(\mathrm{d}\zeta)\mathrm{d}t = \left\langle \alpha(t), \nu^i(\mathrm{d}\zeta)\right\rangle \mathrm{d}t, \tag{2.4}$$

where $\nu^i(\mathrm{d}\zeta) := (\nu^i_{e_1}(\mathrm{d}\zeta), \nu^i_{e_2}(\mathrm{d}\zeta), \ldots, \nu^i_{e_D}(\mathrm{d}\zeta))^\top \in \mathbb{R}^D$. Here we use the subscript α in $n^i_\alpha, i = 1, \ldots, M$, to indicate the dependence of the probability law of the Poisson random measure on the Markov chain $\alpha(t)$. Indeed, for each $j = 1, \ldots, D$, $\nu^i_{e_j}(\mathrm{d}\zeta)$ is the conditional Lévy density of jump sizes of the random measure $N^i(\mathrm{d}t, \mathrm{d}\zeta)$ when $\alpha(t) = e_j$. Moreover, denote $\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}\zeta)$ by

$$\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}\zeta) := (N^1(\mathrm{d}t, \mathrm{d}\zeta) - n^1_\alpha(\mathrm{d}t, \mathrm{d}\zeta), \ldots, N^M(\mathrm{d}t, \mathrm{d}\zeta) - n^M_\alpha(\mathrm{d}t, \mathrm{d}\zeta))^\top. \tag{2.5}$$

Let U be a nonempty subset of some Euclidean space and $u(t) = u(t,\omega) : [0,T] \times \Omega \to U$ be a control process. In what follows, we will set

$$\lambda(t) := (\lambda_1(t), \dots, \lambda_D(t))^\top \text{ and } \nu_\alpha(\mathrm{d}\zeta) := (\nu_\alpha^1(\mathrm{d}\zeta), \dots, \nu_\alpha^M(\mathrm{d}\zeta))^\top.$$

Denote by $\mathcal{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu_\alpha; \mathbb{R}^M)$ the set of square integrable functions $k(\cdot) : \mathbb{R}_0 \to \mathbb{R}^M$ such that $||k(\cdot)||^2_{\mathcal{L}^2} := \sum_{j=1}^M \int_{\mathbb{R}_0} |k_j(\zeta)|^2 \nu_\alpha^j(\mathrm{d}\zeta) < \infty$ and $\mathcal{M}^2(\mathbb{R}_+; \mathbb{R}^D)$ the set of functions $a(\cdot) : \mathbb{R}_+ \to \mathbb{R}^D$ such that $||a(t)||^2_{\mathcal{M}^2} := \sum_{j=1}^D |a_j(t)|^2 \lambda_j(t) < \infty$.

The partially observed risk-sensitive optimal control problem is stated as follows:

Consider the system:

$$\begin{cases} \mathrm{d}x(t) = b(t, x(t), u(t), \alpha(t))\mathrm{d}t + \sigma(t, x(t), u(t), \alpha(t))\mathrm{d}W(t) \\ \qquad\qquad +\beta(t, x(t), u(t), \alpha(t))\mathrm{d}B(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \eta(t, x(t-), u(t-), \alpha(t-), \zeta)\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}\zeta) \\ \qquad\qquad +\gamma(t, x(t-), u(t-), \alpha(t-))\mathrm{d}\widetilde{\Phi}(t), \\ x(0) = x_0, \qquad t \in [0,T], \end{cases} \tag{2.6}$$

and the observation:

$$\begin{cases} \mathrm{d}Y(t) = h(t, x(t), u(t), \alpha(t))\mathrm{d}t + \mathrm{d}B(t), \\ Y(0) = 0. \qquad t \in [0,T]. \end{cases} \tag{2.7}$$

Here $b : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^L, \sigma : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L\times N}, \beta : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L\times K}, \eta : [0,T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}_0 \to \mathbb{R}^{L\times M}, \gamma : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L\times D}$, $h : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^K$ are given continuous, deterministic and measurable functions, $(W(t), Y(t)) := (W_1(t), \dots, W_N(t), Y_1(t), \dots, Y_K(t))^\top$ is an $N + K$-dimensional standard Brownian motion, $\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}\zeta)$ is an M-dimensional Markov regime-switching random measure defined by (2.5), $\widetilde{\Phi}(t) := (\widetilde{\Phi}_1(t), \dots, \widetilde{\Phi}_D(t))^\top$ with $\widetilde{\Phi}_j(t), j = 1, \dots, D$, defined by (2.3), and x_0 is an $\mathcal{F}_0$-measurable random variable with the law P_0 and independent of $(W(\cdot), Y(\cdot))$. We also assume x_0 has finite moments of arbitrary order.

Putting (2.7) into (2.6), we have (set $\hat{b} = b - \beta h$)

$$\begin{cases} \mathrm{d}x(t) = \hat{b}(t, x(t), u(t), \alpha(t))\mathrm{d}t + \sigma(t, x(t), u(t), \alpha(t))\mathrm{d}W(t) \\ \qquad\qquad +\beta(t, x(t), u(t), \alpha(t))\mathrm{d}Y(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \eta(t, x(t-), u(t-), \alpha(t-), \zeta)\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}\zeta) \\ \qquad\qquad +\gamma(t, x(t-), u(t-), \alpha(t-))\mathrm{d}\widetilde{\Phi}(t), \\ x(0) = x_0, \qquad t \in [0,T]. \end{cases} \tag{2.8}$$

Now we are ready to introduce admissible controls. Let $\mathcal{F}_t^Y$ be the P-completed natural filtration generated by $Y(\cdot)$. We say that $u(\cdot)$ is an admissible control, if it is $\mathcal{F}_t^Y$-predictable and satisfies $\sup_{t\in[0,T]} E|u(t)|^m < \infty$, $m = 1, 2, \dots$. We denote by $\mathcal{U}[0,T]$ the set of all admissible controls.

We now make further assumptions on the above functions.

$(\mathcal{A}1)$ The functions $b, \sigma, \beta, \eta, \gamma, h$ are twice continuously differentiable with respect to x, they and their partial derivatives in x are continuous in (x, u), and for some constant C,

$$\begin{aligned} &(1 + |x| + |u|)^{-1}|b(t, x, u, e_i)| + |b_x(t, x, u, e_i)| + |b^k_{xx}(t, x, u, e_i)| \le C, \\ &(1 + |x| + |u|)^{-1}|\sigma^j(t, x, u, e_i)| + |\sigma^j_x(t, x, u, e_i)| + |\sigma^{j,k}_{xx}(t, x, u, e_i)| \le C, \ j = 1, \dots, N, \\ &|\beta^j(t, x, u, e_i)| + |\beta^j_x(t, x, u, e_i)| + |\beta^{j,k}_{xx}(t, x, u, e_i)| \le C, \ j = 1, \dots, K, \\ &|h(t, x, u, e_i)| + |h_x(t, x, u, e_i)| + |h^k_{xx}(t, x, u, e_i)| \le C, \\ &(1 + |x|^2 + |u|^2)^{-1} \int_{\mathbb{R}_0} |\eta^j(t, x, u, e_i, \zeta)|^2 \nu^j_{e_i}(\mathrm{d}\zeta) + \int_{\mathbb{R}_0} |\eta^j_x(t, x, u, e_i, \zeta)|^2 \nu^j_{e_i}(\mathrm{d}\zeta) \\ &\qquad + \int_{\mathbb{R}_0} |\eta^{j,k}_{xx}(t, x, u, e_i, \zeta)|^2 \nu^j_{e_i}(\mathrm{d}\zeta) \le C, \ j = 1, \dots, M, \\ &(1 + |x|^2 + |u|^2)^{-1}|\gamma^j(t, x, u, e_i)|^2 \lambda_j(t) + |\gamma^j_x(t, x, u, e_i)|^2 \lambda_j(t) \\ &\qquad + |\gamma^{j,k}_{xx}(t, x, u, e_i)|^2 \lambda_j(t) \le C, \ j = 1, \dots, D, \end{aligned}$$

where $k = 1, \dots, L$, σ^j is the jth column of the matrix σ, $\sigma^{j,k}$ is the kth coordinate of vector σ^j, and similar notations have been used for b, β, h, η and γ.

For each $u(\cdot) \in \mathcal{U}[0,T]$, Assumption ($\mathcal{A}1$) implies that (2.8) has a unique strong solution $x(\cdot)$. Define $\mathrm{d}P^u = Z(t)\mathrm{d}P$ with

$$Z(t) := \exp\bigg\{ \int_0^t h^\top(s, x(s), u(s), \alpha(s))\mathrm{d}Y(s) - \frac{1}{2}\int_0^t |h(s, x(s), u(s), \alpha(s))|^2 \mathrm{d}s \bigg\}. \tag{2.9}$$

Obviously, $Z(\cdot)$ can be characterized as the solution to

$$\mathrm{d}Z(t) = Z(t)h^\top(t, x(t), u(t), \alpha(t))\mathrm{d}Y(t), \qquad Z(0) = 1. \tag{2.10}$$

Then Girsanov's theorem and Assumption ($\mathcal{A}1$) imply that P^u is a new probability measure under which $(W(\cdot), B(\cdot))$ is an $N + K$-dimensional standard Brownian motion, $\widetilde{N}_\alpha(\cdot,\cdot)$ is still a compensated Poisson random measure and $\widetilde{\Phi}(\cdot)$ is still a martingale.

We first introduce the performance criterion in risk-sensitive controls. Given a monotonically increasing function $\Psi : \mathbb{R} \to \mathbb{R}$ (Ψ is called "disutility function"), the objective is to find an admissible control $u(\cdot)$ so as to minimize

$$J(x_0, e_i; u(\cdot)) := E^u\Psi\bigg[\int_0^T f(t, x(t), u(t), \alpha(t))\mathrm{d}t + g(x(T), \alpha(T)) \bigg], \tag{2.11}$$

where E^u denotes the expectation with respect to the probability measure P^u, f and g are suitable functions such that $J(x_0, e_i; u(\cdot)) > -\infty$ for any $u(\cdot) \in \mathcal{U}[0,T]$. In the above, the function Ψ is usually taken to be either concave or convex, which corresponds to the risk-seeking or risk-averse attitude of the controller. Clearly, the cost functional (2.11) subject to (2.8) and (2.10) constitutes a partially observed risk-sensitive optimal control problem. Let us elaborate the meaning of risk-sensitive by an intuitive argument. We also refer to [30] for more details on risk-sensitive controls. Define $X = \int_0^T f(t, x(t), u(t), \alpha(t))\mathrm{d}t + g(x(T), \alpha(T))$ and suppose Ψ is differentiable at $E^u[X]$. Then using Taylor's expansion yields

$$\Psi(X) \approx \Psi(E^u[X]) + \Psi'(E^u[X])(X - E^u[X]) + \frac{1}{2}\Psi''(E^u[X])(X - E^u[X])^2.$$

If Ψ is strictly concave near $E^u[X]$, then $\Psi''(E^u[X]) < 0$. This will reduce the overall cost with a large $|X - E^u[X]|$, which implies the controller is risk-seeking. Conversely, if Ψ is strictly convex near $E^u[X]$, then $\Psi''(E^u[X]) > 0$. This will introduce a penalty to the variance term $(X - E^u[X])^2$ in the overall cost. In this case, the controller tries to avoid a large deviation of X from its mean $E^u[X]$, which implies the controller is risk-averse. Finally, if $\Psi''(E^u[X])$ is close or equal to 0, then $E^u[\Psi(X)] \approx \Psi(E^u[X])$, in which case the risk-sensitive model reduces to the risk-neutral one.

A commonly used and extensively studied function for Ψ is the constant absolute risk aversion (CARA) utility or exponential utility

$$\Psi(x) = \theta e^{\theta x}, \quad \theta \neq 0, \tag{2.12}$$

where $\theta \in \mathbb{R}$ is a fixed constant representing the risk sensitivity degree of the criterion. $\theta < 0$ and $\theta > 0$ correspond to risk-seeking and risk-averse situations, respectively. In particular, when θ is sufficiently small, the problem can be well approximated by its risk-neutral counterpart. Risk-sensitive optimal control problems with the disutility function of the form (2.12) have been discussed by many researchers. The main reason is that such model seems more practical and can be used to study many financial problems. See, for example, [13, 14, 20, 33] and references therein.

In this paper, as an extension of the expected CARA utility maximization problem, we consider the following cost functional:

$$J(x_0, e_i; u(\cdot)) := E^u\Big[e^{\theta\big(g(x(T),\alpha(T)) + \int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\big)} \Big], \tag{2.13}$$

where $f : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R},\ g : \mathbb{R}^L \times S \to \mathbb{R}$ are Borel measurable, continuous in u and twice continuously differentiable in x, and $\theta > 0$, the risk-sensitive parameter, is a fixed constant.

Our partially observed risk-sensitive optimal control problem is to minimize the cost functional (2.13) over $u(\cdot) \in \mathcal{U}[0,T]$ subject to (2.8) and (2.10). Obviously, (2.13) can be rewritten as

$$J(x_0, e_i; u(\cdot)) := E\Big[Z(T)e^{\theta\left(g(x(T),\alpha(T))+\int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\right)}\Big]. \tag{2.14}$$

Thus, the original problem (2.13) is equivalent to minimizing (2.14) subject to (2.8) and (2.10).

Remark 2.1. *The present formulation of the partially observed risk-sensitive optimal control problem is quite similar to the completely observed case. The only difference lies in the admissible class $\mathcal{U}[0,T]$ of controls are partially observed.*

For each $e_i \in S$, we also make the following assumptions.

($\mathcal{A}2$) There exists a constant $C > 0$ such that

$$(1+|x|+|u|^2)^{-1}|f(t,x,u,e_i)| + |f_x(t,x,u,e_i)| + |f_{xx}(t,x,u,e_i)| \le C,$$
$$(1+|x|)^{-1}|g(x,e_i)| + |g_x(x,e_i)| + |g_{xx}(x,e_i)| \le C.$$

($\mathcal{A}3$) $E\Big[e^{2\theta\left(|g(x(T),\alpha(T))|+\int_0^T |f(t,x(t),u(t),\alpha(t))|\mathrm{d}t\right)}\Big] < +\infty$ holds.

3. Partial Information Maximum Principle

In this section, we combine Girsanov's theorem with a standard spike variational technique to derive two general maximum principles for the partially observed risk-sensitive optimal control problem (2.14) subject to (2.8) and (2.10).

Let $\bar{u}(\cdot)$ be an optimal control, $\bar{x}(\cdot)$ and $\bar{Z}(\cdot)$ be the corresponding solution to (2.8) and (2.10), respectively. Now we introduce the spike variation of the control $\bar{u}(\cdot)$ as follows.

Let $u(\cdot) \in \mathcal{U}[0,T]$ be any given control and $\varepsilon > 0$. Define

$$u^\varepsilon(t) = \begin{cases} \bar{u}(t), & t \in [0,T]\backslash E_\varepsilon, \\ u(t), & t \in E_\varepsilon, \end{cases} \tag{3.1}$$

where $E_\varepsilon \subset [0,T]$ is a measurable set with its Lebesgue measure $|E_\varepsilon| = \varepsilon$. We refer to $u^\varepsilon(\cdot)$ as a spike variation of the control $\bar{u}(\cdot)$. Let $x^\varepsilon(\cdot)$ and $Z^\varepsilon(\cdot)$ be the solution to (2.8) and (2.10) under the control $u^\varepsilon(\cdot)$ respectively.

For simplification, we introduce the following notations:

$$\begin{cases} \varphi_x(t) \triangleq \varphi_x(t,\bar{x}(t),\bar{u}(t),\alpha(t)),\ \ \varphi_{xx}(t) \triangleq \varphi_{xx}(t,\bar{x}(t),\bar{u}(t),\alpha(t)), \\ \eta_x^j(t,\zeta) \triangleq \eta_x^j(t,\bar{x}(t-),\bar{u}(t-),\alpha(t-),\zeta),\ \ \eta_{xx}^{j,k}(t,\zeta) \triangleq \eta_{xx}^{j,k}(t,\bar{x}(t-),\bar{u}(t-),\alpha(t-),\zeta), \\ \gamma_x^j(t) \triangleq \gamma_x^j(t,\bar{x}(t-),\bar{u}(t-),\alpha(t-)),\ \ \gamma_{xx}^{j,k}(t) \triangleq \gamma_{xx}^{j,k}(t,\bar{x}(t-),\bar{u}(t-),\alpha(t-)), \\ \delta\varphi(t,u) \triangleq \varphi(t,\bar{x}(t-),u,\alpha(t-)) - \varphi(t,\bar{x}(t-),\bar{u}(t),\alpha(t-)), \\ \delta\eta(t,u,\zeta) \triangleq \eta(t,\bar{x}(t-),u,\alpha(t-),\zeta) - \eta(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\zeta), \\ \delta\gamma(t,u) \triangleq \gamma(t,\bar{x}(t-),u,\alpha(t-)) - \gamma(t,\bar{x}(t-),\bar{u}(t),\alpha(t-)), \end{cases} \tag{3.2}$$

where $\varphi = b^k, \sigma^{j,k}, \beta^{j,k}, h^k, \hat{b}^k, f$.

We now introduce the first-order variational equation

$$\begin{cases} \mathrm{d}x_1(t) = \hat{b}_x(t)x_1(t)\mathrm{d}t + \sum\limits_{j=1}^N \Big\{\sigma_x^j(t)x_1(t) + \delta\sigma^j(t,u(t))I_{E_\varepsilon}(t)\Big\}\mathrm{d}W_j(t) \\ \qquad\qquad + \sum\limits_{j=1}^K \Big\{\beta_x^j(t)x_1(t) + \delta\beta^j(t,u(t))I_{E_\varepsilon}(t)\Big\}\mathrm{d}Y_j(t) \\ \qquad\qquad + \sum\limits_{j=1}^M \int_{\mathbb{R}_0} \Big\{\eta_x^j(t,\zeta)x_1(t-) + \delta\eta^j(t,u(t),\zeta)I_{E_\varepsilon}(t)\Big\}\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}\zeta) \\ \qquad\qquad + \sum\limits_{j=1}^D \Big\{\gamma_x^j(t)x_1(t-) + \delta\gamma^j(t,u(t))I_{E_\varepsilon}(t)\Big\}\mathrm{d}\widetilde{\Phi}_j(t), \\ \mathrm{d}Z_1(t) = \Big[Z_1(t)h(t) + \bar{Z}(t)h_x(t)x_1(t) + \bar{Z}(t)\delta h(t,u(t))I_{E_\varepsilon}(t)\Big]^\top \mathrm{d}Y(t), \\ x_1(0) = 0,\ Z_1(0) = 0,\ t \in [0,T], \end{cases} \tag{3.3}$$

and the second-order variational equation

$$\begin{cases} \mathrm{d}x_2(t) = \Big\{\hat{b}_x(t)x_2(t) + \delta\hat{b}(t,u(t))I_{E_\varepsilon}(t) + \frac{1}{2}\hat{b}_{xx}(t)x_1(t)^2\Big\}\mathrm{d}t \\ \qquad + \sum_{j=1}^{N}\Big\{\sigma_x^j(t)x_2(t) + \delta\sigma_x^j(t,u(t))x_1(t)I_{E_\varepsilon}(t) + \frac{1}{2}\sigma_{xx}^j(t)x_1(t)^2\Big\}\mathrm{d}W_j(t) \\ \qquad + \sum_{j=1}^{K}\Big\{\beta_x^j(t)x_2(t) + \delta\beta_x^j(t,u(t))x_1(t)I_{E_\varepsilon}(t) + \frac{1}{2}\beta_{xx}^j(t)x_1(t)^2\Big\}\mathrm{d}Y_j(t) \\ \qquad + \sum_{j=1}^{M}\int_{\mathbb{R}_0}\Big\{\eta_x^j(t,\zeta)x_2(t-) + \delta\eta_x^j(t,u(t),\zeta)x_1(t-)I_{E_\varepsilon}(t) \\ \qquad\qquad + \frac{1}{2}\eta_{xx}^j(t,\zeta)x_1(t-)^2\Big\}\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}\zeta) \\ \qquad + \sum_{j=1}^{D}\Big\{\gamma_x^j(t)x_2(t-) + \delta\gamma_x^j(t,u(t))x_1(t-)I_{E_\varepsilon}(t) + \frac{1}{2}\gamma_{xx}^j(t)x_1(t-)^2\Big\}\mathrm{d}\widetilde{\Phi}_j(t), \\ \mathrm{d}Z_2(t) = \Big[Z_2(t)h(t) + Z_1(t)h_x(t)x_1(t) + Z_1(t)\delta h(t)I_{E_\varepsilon}(t) + \bar{Z}(t)h_x(t)x_2(t) \\ \qquad + \frac{1}{2}\bar{Z}(t)h_{xx}(t)x_1(t)^2 + \bar{Z}(t)\delta h_x(t)x_1(t)I_{E_\varepsilon}(t)\Big]^\top \mathrm{d}Y(t), \\ x_2(0) = 0,\ Z_2(0) = 0,\ t \in [0,T]. \end{cases} \tag{3.4}$$

Here, we are using the notation $h(t) = h(t,\bar{x}(t),\bar{u}(t),\alpha(t))$,

$$\hat{b}_{xx}(t)x_1(t)^2 := \begin{pmatrix} \mathrm{tr}\big[\hat{b}_{xx}^1(t)x_1(t)x_1(t)^\top\big] \\ \vdots \\ \mathrm{tr}\big[\hat{b}_{xx}^L(t)x_1(t)x_1(t)^\top\big] \end{pmatrix}$$

and similarly for $\sigma_{xx}^j(t)x_1(t)^2, \beta_{xx}^j(t)x_1(t)^2, \eta_{xx}^j(t,\zeta)x_1(t-)^2, \gamma_{xx}^j(t)x_1(t-)^2$. Under Assumption $(\mathcal{A}1)$, both (3.3) and (3.4) admit unique $\mathbb{F}$-adapted càdlàg (i.e. right continuous with left limits) solutions. By a linear method similar to [24], we have the following lemmas.

Lemma 3.1. *Let Assumption* $(\mathcal{A}1)$ *hold. For any* $u(\cdot) \in \mathcal{U}[0,T]$ *and* $k \geq 1$*, the solutions of* (2.8) *and* (2.10) *satisfy*

$$\sup_{t\in[0,T]} E|Z(t)|^{2k} < +\infty \text{ and } \sup_{t\in[0,T]} E|x(t)|^{2k} \leq C\Big(1 + \sup_{t\in[0,T]} E|u(t)|^{2k}\Big). \tag{3.5}$$

Proof. In view of Assumption $(\mathcal{A}1)$, the desired results follow from the application of Burkholder-Davis-Gundy (B-D-G) inequality and Gronwall's inequality. □

Lemma 3.2. *Let Assumption* $(\mathcal{A}1)$ *hold. Then for any* $k \geq 1$,

$$\begin{aligned} &\sup_{t\in[0,T]} E|x_1(t)|^{2k} = O(\varepsilon^k),\ \sup_{t\in[0,T]} E|Z_1(t)|^{2k} = O(\varepsilon^k), \\ &\sup_{t\in[0,T]} E|x_2(t)|^{2k} = O(\varepsilon^{2k}),\ \sup_{t\in[0,T]} E|Z_2(t)|^{2k} = O(\varepsilon^{2k}), \\ &\sup_{t\in[0,T]} E|x^\varepsilon(t) - \bar{x}(t) - x_1(t) - x_2(t)|^{2k} = o(\varepsilon^{2k}), \\ &\sup_{t\in[0,T]} E|Z^\varepsilon(t) - \bar{Z}(t) - Z_1(t) - Z_2(t)|^{2k} = o(\varepsilon^{2k}). \end{aligned} \tag{3.6}$$

Proof. First of all, we state some general estimates, which will be frequently used below. Let $f_0(\cdot)$ and $g_0(\cdot,\cdot)$ be processes such that the expressions involving them below make sense. Then, we can easily verify the following estimates:

$$\begin{aligned} &E\bigg|\int_0^T f_0(t)I_{E_\varepsilon}(t)\mathrm{d}t\bigg|^p \leq C\varepsilon^{p-1}E\int_{E_\varepsilon}|f_0(t)|^p\mathrm{d}t, \\ &E\bigg|\int_0^T f_0(t)I_{E_\varepsilon}(t)\mathrm{d}V(t)\bigg|^{2p} \leq C\varepsilon^{p-1}E\int_{E_\varepsilon}|f_0(t)|^{2p}\mathrm{d}t, \\ &E\bigg|\int_0^T f_0(t)I_{E_\varepsilon}(t)\mathrm{d}\widetilde{\Phi}_j(t)\bigg|^{2p} \leq C\varepsilon^{p-1}E\int_{E_\varepsilon}|f_0(t)^2\lambda_j(t)|^p\mathrm{d}t, \end{aligned} \tag{3.7}$$

$$E\bigg|\int_0^T\int_{\mathbb{R}_0} g_0(t,\zeta)I_{E_\varepsilon}(t)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}\zeta)\bigg|^{2p}\leq C\varepsilon^{p-1}E\int_{E_\varepsilon}\bigg|\int_{\mathbb{R}_0}|g_0(t,\zeta)|^2\nu_\alpha^j(\mathrm{d}\zeta)\bigg|^p\mathrm{d}t,$$

with $p>1$ and $V(\cdot)=W(\cdot)$ or $Y(\cdot)$.

By virtue of Assumption $(\mathcal{A}1)$, Lemma 3.1 and (3.7), we have

$$\begin{aligned}
&E\bigg|\int_0^t \hat{b}_x(s)x_1(s)\mathrm{d}s\bigg|^{2k}\leq CE\int_0^t|x_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t \sigma_x^j(s)x_1(s)\mathrm{d}W_j(s)\bigg|^{2k}\leq CE\int_0^t|x_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t \delta\sigma^j(s,u(s))I_{E_\varepsilon}(s)\mathrm{d}W_j(s)\bigg|^{2k}\leq C\varepsilon^{k-1}E\int_{E_\varepsilon}|\delta\sigma^j(s,u(s))|^{2k}\mathrm{d}s\leq C\varepsilon^k,\\
&E\bigg|\int_0^t \beta_x^j(s)x_1(s)\mathrm{d}Y_j(s)\bigg|^{2k}\leq CE\int_0^t|x_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t \delta\beta^j(s,u(s))I_{E_\varepsilon}(s)\mathrm{d}Y_j(s)\bigg|^{2k}\leq C\varepsilon^{k-1}E\int_{E_\varepsilon}|\delta\beta^j(s,u(s))|^{2k}\mathrm{d}s\leq C\varepsilon^k,\\
&E\bigg|\int_0^t\int_{\mathbb{R}_0} \eta_x^j(s,\zeta)x_1(s-)\widetilde{N}_\alpha^j(\mathrm{d}s,\mathrm{d}\zeta)\bigg|^{2k}\leq CE\int_0^t|x_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t\int_{\mathbb{R}_0} \delta\eta^j(s,u(s),\zeta)I_{E_\varepsilon}(s)\widetilde{N}_\alpha^j(\mathrm{d}s,\mathrm{d}\zeta)\bigg|^{2k}\\
&\qquad\qquad\leq C\varepsilon^{k-1}E\int_{E_\varepsilon}\bigg|\int_{\mathbb{R}_0}|\delta\eta^j(s,u(s),\zeta)|^2\nu_\alpha^j(\mathrm{d}\zeta)\bigg|^k\mathrm{d}s\leq C\varepsilon^k,\\
&E\bigg|\int_0^t \gamma_x^j(s)x_1(s-)\mathrm{d}\widetilde{\Phi}_j(s)\bigg|^{2k}\leq CE\int_0^t|x_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t \delta\gamma^j(s,u(s))I_{E_\varepsilon}(s)\mathrm{d}\widetilde{\Phi}_j(s)\bigg|^{2k}\leq C\varepsilon^{k-1}E\int_{E_\varepsilon}|\delta\gamma^j(s,u(s))^2\lambda_j(t)|^k\mathrm{d}s\leq C\varepsilon^k,
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&E\bigg|\int_0^t\Big(Z_1(s)h(s)\Big)^\top\mathrm{d}Y(s)\bigg|^{2k}\leq CE\int_0^t|Z_1(s)|^{2k}\mathrm{d}s,\\
&E\bigg|\int_0^t\Big(\bar{Z}(s)h_x(s)x_1(s)\Big)^\top\mathrm{d}Y(s)\bigg|^{2k}\leq CE\int_0^t|\bar{Z}(s)x_1(s)|^{2k}\mathrm{d}s\\
&\qquad\leq C\sqrt{E\int_0^t|x_1(s)|^{4k}\mathrm{d}s}\leq C\sqrt{\sup_{t\in[0,T]}E|x_1(s)|^{4k}},\\
&E\bigg|\int_0^t\Big(\bar{Z}(s)\delta h(s,u(s))I_{E_\varepsilon}(s)\Big)^\top\mathrm{d}Y(s)\bigg|^{2k}\leq C\varepsilon^{k-1}E\int_{E_\varepsilon}|\bar{Z}(s)\delta h(s,u(s))|^{2k}\mathrm{d}s\leq C\varepsilon^k.
\end{aligned}\tag{3.9}$$

Then by using the familiar elementary inequality

$$|m_1+m_2+\ldots+m_k|^p\leq C(|m_1|^p+|m_2|^p+\ldots+|m_k|^p),\qquad\forall p\in\mathbb{N},\ k\geq1,\tag{3.10}$$

we have

$$E|x_1(t)|^{2k}\leq C\bigg\{E\int_0^t|x_1(s)|^{2k}\mathrm{d}s+\varepsilon^k\bigg\},\tag{3.11}$$

and

$$E|Z_1(t)|^{2k}\leq C\bigg\{E\int_0^t|Z_1(s)|^{2k}\mathrm{d}s+\sqrt{\sup_{t\in[0,T]}E|x_1(s)|^{4k}}+\varepsilon^k\bigg\}.\tag{3.12}$$

Hence, applying Gronwall's inequality, we derive the first two estimates of (3.6). In a similar way, the third and fourth estimates of (3.6) can also be obtained.

The proof for the fifth estimate is giving in the following. Set $\Delta x(t) := x_1(t) + x_2(t)$, we have

$$\begin{aligned}
&\int_0^t \hat{b}(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}s = \int_0^t \Big[\hat{b}(s,\bar{x},u^\varepsilon,\alpha(s)) + \hat{b}_x(s,\bar{x},u^\varepsilon,\alpha(s))\Delta x \\
&\qquad + \sum_{k,r=1}^{L}\int_0^1\int_0^1 \lambda \hat{b}_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k\Delta x^r\Big]\mathrm{d}s,\\
&\int_0^t \sigma^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}W_j(s) = \int_0^t \Big[\sigma^j(s,\bar{x},u^\varepsilon,\alpha(s)) + \sigma^j_x(s,\bar{x},u^\varepsilon,\alpha(s))\Delta x \\
&\qquad + \sum_{k,r=1}^{L}\int_0^1\int_0^1 \lambda \sigma^j_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k\Delta x^r\Big]\mathrm{d}W_j(s),\\
&\int_0^t \beta^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}Y_j(s) = \int_0^t \Big[\beta^j(s,\bar{x},u^\varepsilon,\alpha(s)) + \beta^j_x(s,\bar{x},u^\varepsilon,\alpha(s))\Delta x \\
&\qquad + \sum_{k,r=1}^{L}\int_0^1\int_0^1 \lambda \beta^j_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k\Delta x^r\Big]\mathrm{d}Y_j(s),\\
&\int_0^t\int_{\mathbb{R}_0} \eta^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s),\zeta)\widetilde{N}^j_\alpha(\mathrm{d}s,\mathrm{d}\zeta) = \int_0^t\int_{\mathbb{R}_0} \Big[\eta^j(s,\bar{x},u^\varepsilon,\alpha(s),\zeta) + \eta^j_x(s,\bar{x},u^\varepsilon,\alpha(s),\zeta)\Delta x \\
&\qquad + \sum_{k,r=1}^{L}\int_0^1\int_0^1 \lambda \eta^j_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s),\zeta)\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k\Delta x^r\Big]\widetilde{N}^j_\alpha(\mathrm{d}s,\mathrm{d}\zeta),\\
&\int_0^t \gamma^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\widetilde{\Phi}_j(s) = \int_0^t \Big[\gamma^j(s,\bar{x},u^\varepsilon,\alpha(s)) + \gamma^j_x(s,\bar{x},u^\varepsilon,\alpha(s))\Delta x \\
&\qquad + \sum_{k,r=1}^{L}\int_0^1\int_0^1 \lambda \gamma^j_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k\Delta x^r\Big]\mathrm{d}\widetilde{\Phi}_j(s).
\end{aligned}\tag{3.13}$$

From the above equalities, we get

$$\begin{aligned}
&\bar{x}(t)+\Delta x(t)-x_0+\int_0^t A^1_\varepsilon(s)\mathrm{d}s+\sum_{j=1}^N\int_0^t A^{2,j}_\varepsilon(s)\mathrm{d}W_j(s)+\sum_{j=1}^K\int_0^t A^{3,j}_\varepsilon(s)\mathrm{d}Y_j(s)\\
&\quad+\sum_{j=1}^M\int_0^t\int_{\mathbb{R}_0} A^{4,j}_\varepsilon(s,\zeta)\widetilde{N}^j_\alpha(\mathrm{d}s,\mathrm{d}\zeta)+\sum_{j=1}^D\int_0^t A^{5,j}_\varepsilon(s)\mathrm{d}\widetilde{\Phi}_j(s)\\
&=\int_0^t \hat{b}(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}s+\sum_{j=1}^N\int_0^t \sigma^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}W_j(s)\\
&\quad+\sum_{j=1}^K\int_0^t \beta^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}Y_j(s)+\sum_{j=1}^D\int_0^t \gamma^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s))\mathrm{d}\widetilde{\Phi}_j(s)\\
&\quad+\sum_{j=1}^M\int_0^t\int_{\mathbb{R}_0} \eta^j(s,\bar{x}+\Delta x,u^\varepsilon,\alpha(s),\zeta)\widetilde{N}^j_\alpha(\mathrm{d}s,\mathrm{d}\zeta),
\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}
A^1_\varepsilon(s) &= \delta\hat{b}_x(s)I_{E_\varepsilon}(s)\Delta x(s)+\sum_{k,r=1}^L\Big\{\frac{1}{2}\hat{b}_{x^k x^r}(s)[x^k_2(s)x^r_2(s)+2x^k_1(s)x^r_2(s)]\\
&\quad+\int_0^1\int_0^1\lambda\Big[\hat{b}_{x^k x^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))-\hat{b}_{x^k x^r}(s)\Big]\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k(s)\Delta x^r(s)\Big\},
\end{aligned}$$

$$
\begin{aligned}
A_\varepsilon^{2,j}(s) &= \delta\sigma_x^j(s)I_{E_\varepsilon}(s)x_2(s) + \sum_{k,r=1}^{L}\bigg\{\frac{1}{2}\sigma^j_{x^kx^r}(s)[x_2^k(s)x_2^r(s)+2x_1^k(s)x_2^r(s)]\\
&\quad+\int_0^1\int_0^1\lambda\Big[\sigma^j_{x^kx^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))-\sigma^j_{x^kx^r}(s)\Big]\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k(s)\Delta x^r(s)\bigg\},\\
A_\varepsilon^{3,j}(s) &= \delta\beta_x^j(s)I_{E_\varepsilon}(s)x_2(s) + \sum_{k,r=1}^{L}\bigg\{\frac{1}{2}\beta^j_{x^kx^r}(s)[x_2^k(s)x_2^r(s)+2x_1^k(s)x_2^r(s)]\\
&\quad+\int_0^1\int_0^1\lambda\Big[\beta^j_{x^kx^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))-\beta^j_{x^kx^r}(s)\Big]\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k(s)\Delta x^r(s)\bigg\},\\
A_\varepsilon^{4,j}(s,\zeta) &= \delta\eta_x^j(s,\zeta)I_{E_\varepsilon}(s)x_2(s) + \sum_{k,r=1}^{L}\bigg\{\frac{1}{2}\eta^j_{x^kx^r}(s,\zeta)[x_2^k(s)x_2^r(s)+2x_1^k(s)x_2^r(s)]\\
&\quad+\int_0^1\int_0^1\lambda\Big[\eta^j_{x^kx^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s),\zeta)-\eta^j_{x^kx^r}(s,\zeta)\Big]\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k(s)\Delta x^r(s)\bigg\},\\
A_\varepsilon^{5,j}(s) &= \delta\gamma_x^j(s)I_{E_\varepsilon}(s)x_2(s) + \sum_{k,r=1}^{L}\bigg\{\frac{1}{2}\gamma^j_{x^kx^r}(s)[x_2^k(s)x_2^r(s)+2x_1^k(s)x_2^r(s)]\\
&\quad+\int_0^1\int_0^1\lambda\Big[\gamma^j_{x^kx^r}(s,\bar{x}+\lambda\mu\Delta x,u^\varepsilon,\alpha(s))-\gamma^j_{x^kx^r}(s)\Big]\mathrm{d}\lambda\mathrm{d}\mu\Delta x^k(s)\Delta x^r(s)\bigg\}. \qquad (3.15)
\end{aligned}
$$

It follows from (3.14) that

$$
\begin{aligned}
(x^\varepsilon-\bar{x}-\Delta x)(t) \;=\; &\int_0^t\big[\tilde{A}_\varepsilon^1(s)(x^\varepsilon-\bar{x}-\Delta x)(s)+A_\varepsilon^1(s)\big]\mathrm{d}s\\
&+\sum_{j=1}^{N}\int_0^t\big[\tilde{A}_\varepsilon^{2,j}(s)(x^\varepsilon-\bar{x}-\Delta x)(s)+A_\varepsilon^{2,j}(s)\big]\mathrm{d}W_j(s)\\
&+\sum_{j=1}^{K}\int_0^t\big[\tilde{A}_\varepsilon^{3,j}(s)(x^\varepsilon-\bar{x}-\Delta x)(s)+A_\varepsilon^{3,j}(s)\big]\mathrm{d}Y_j(s)\\
&+\sum_{j=1}^{M}\int_0^t\int_{\mathbb{R}_0}\big[\tilde{A}_\varepsilon^{4,j}(s,\zeta)(x^\varepsilon-\bar{x}-\Delta x)(s)+A_\varepsilon^{4,j}(s,\zeta)\big]\widetilde{N}_\alpha^j(\mathrm{d}s,\mathrm{d}\zeta)\\
&+\sum_{j=1}^{D}\int_0^t\big[\tilde{A}_\varepsilon^{5,j}(s)(x^\varepsilon-\bar{x}-\Delta x)(s)+A_\varepsilon^{5,j}(s)\big]\mathrm{d}\widetilde{\Phi}_j(s), \qquad (3.16)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{A}_\varepsilon^1(s) &= \int_0^1\hat{b}_x(s,\lambda x^\varepsilon+(1-\lambda)(\bar{x}+\Delta x),u^\varepsilon,\alpha(s))\mathrm{d}\lambda,\\
\tilde{A}_\varepsilon^{2,j}(s) &= \int_0^1\sigma_x^j(s,\lambda x^\varepsilon+(1-\lambda)(\bar{x}+\Delta x),u^\varepsilon,\alpha(s))\mathrm{d}\lambda,\\
\tilde{A}_\varepsilon^{3,j}(s) &= \int_0^1\beta_x^j(s,\lambda x^\varepsilon+(1-\lambda)(\bar{x}+\Delta x),u^\varepsilon,\alpha(s))\mathrm{d}\lambda,\\
\tilde{A}_\varepsilon^{4,j}(s,\zeta) &= \int_0^1\eta_x^j(s,\lambda x^\varepsilon+(1-\lambda)(\bar{x}+\Delta x),u^\varepsilon,\alpha(s),\zeta)\mathrm{d}\lambda,\\
\tilde{A}_\varepsilon^{5,j}(s) &= \int_0^1\gamma_x^j(s,\lambda x^\varepsilon+(1-\lambda)(\bar{x}+\Delta x),u^\varepsilon,\alpha(s))\mathrm{d}\lambda. \qquad (3.17)
\end{aligned}
$$

Following Assumption ($\mathcal{A}1$) and noting (3.7) and (3.15), we have

$$|\tilde{A}^1_\varepsilon(s)|+\sum_{j=1}^N|\tilde{A}^{2,j}_\varepsilon(s)|+\sum_{j=1}^K|\tilde{A}^{3,j}_\varepsilon(s)|+\sum_{j=1}^M\int_{\mathbb{R}_0}|\tilde{A}^{4,j}_\varepsilon(s,\zeta)|^2\nu^j_\alpha(\mathrm{d}\zeta)+\sum_{j=1}^D|\tilde{A}^{5,j}_\varepsilon(s)|^2\lambda_j(s)\le C,$$

$$\begin{aligned}\sup_{t\in[0,T]}E\bigg\{\bigg|\int_0^t A^1_\varepsilon(s)\mathrm{d}s\bigg|^{2k}&+\sum_{j=1}^N\bigg|\int_0^t A^{2,j}_\varepsilon(s)\mathrm{d}W_j(s)\bigg|^{2k}+\sum_{j=1}^K\bigg|\int_0^t A^{3,j}_\varepsilon(s)\mathrm{d}Y_j(s)\bigg|^{2k}\\&+\sum_{j=1}^M\bigg|\int_0^t\int_{\mathbb{R}_0}A^{4,j}_\varepsilon(s,\zeta)\widetilde{N}^j_\alpha(\mathrm{d}s,\mathrm{d}\zeta)\bigg|^{2k}+\sum_{j=1}^D\bigg|\int_0^t A^{5,j}_\varepsilon(s)\mathrm{d}\widetilde{\Phi}_j(s)\bigg|^{2k}=o(\varepsilon^{2k}).\end{aligned}\tag{3.18}$$

From (3.18), we can use (3.10) and Gronwall's inequality to obtain the fifth estimate. The last estimate of (3.6) can be proved similarly. □

3.1. **The case of "$f=0$".**

In this subsection, we consider the special case of the cost functional for $f=0$ in (2.14). Then (2.14) becomes

$$J(x_0,e_i;u(\cdot)):=E\Big[Z(T)e^{\theta g(x(T),\alpha(T))}\Big].\tag{3.19}$$

Then we have the following lemma.

Lemma 3.3. *(Variational Inequality) Let Assumptions ($\mathcal{A}1$)-($\mathcal{A}3$) hold. Then we have*

$$\begin{aligned}&E^{\bar u}\Big[\bar Z^{-1}(T)(Z_1(T)+Z_2(T))e^{\theta g(\bar x(T),\alpha(T))}\Big]\\&\quad+\theta E^{\bar u}\Big[\bar Z^{-1}(T)Z_1(T)e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))x_1(T)\Big]\\&\quad+\theta E^{\bar u}\Big[e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))(x_1(T)+x_2(T))\Big]\\&\quad+\frac12\theta^2E^{\bar u}\Big[e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_x(\bar x(T),\alpha(T))g_x^\top(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]\\&\quad+\frac12\theta E^{\bar u}\Big[e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_{xx}(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]\ge o(\varepsilon).\end{aligned}\tag{3.20}$$

Proof. Using the fact that $J(x_0,e_i;u^\varepsilon(\cdot))-J(x_0,e_i;\bar u(\cdot))\ge0$, the Taylor expansion and Lemma 3.2, we have

$$\begin{aligned}0&\le E\Big[Z^\varepsilon(T)e^{\theta g(x^\varepsilon(T),\alpha(T))}\Big]-E\Big[\bar Z(T)e^{\theta g(\bar x(T),\alpha(T))}\Big]\\&=E\Big[(Z_1(T)+Z_2(T))e^{\theta g(\bar x(T),\alpha(T))}\Big]\\&\quad+\theta E\Big[Z_1(T)e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))x_1(T)\Big]\\&\quad+\theta E\Big[\bar Z(T)e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))(x_1(T)+x_2(T))\Big]\\&\quad+\frac12\theta^2E\Big[\bar Z(T)e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_x(\bar x(T),\alpha(T))g_x^\top(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]\\&\quad+\frac12\theta E\Big[\bar Z(T)e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_{xx}(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]+o(\varepsilon).\end{aligned}\tag{3.21}$$

Thus, we obtain the desired conclusion. □

Let $\mathcal{H}$ be a finite-dimensional vector or matrix space. We define

$$L^2_{\mathcal{F}}([0,T];\mathcal{H}):=\bigg\{f:\mathcal{H}\text{-valued }\mathcal{F}_t\text{-adapted processes, s.t. }E\bigg[\int_0^T\bar Z(t)|f(t)|^2\mathrm{d}t\bigg]<\infty\bigg\};$$

$$L^2_{\mathcal{F},p}([0,T];\mathcal{H}):=\bigg\{f:\mathcal{H}\text{-valued }\mathcal{F}_t\text{-predictable processes, s.t. }E\bigg[\int_0^T\bar Z(t)|f(t)|^2\mathrm{d}t\bigg]<\infty\bigg\};$$

$$F^2_p([0,T];\mathcal{H}):=\bigg\{f:\mathcal{H}\text{-valued }\mathcal{F}_t\text{-predictable processes, s.t. }E\bigg[\int_0^T\bar Z(t)||f(t,\cdot)||^2_{\mathcal{L}^2}\mathrm{d}t\bigg]<\infty\bigg\};$$

$$M_p^2([0,T];\mathcal{H}) := \Big\{ f : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-predictable processes, s.t. } E\Big[\int_0^T \bar{Z}(t)||f(t)||^2_{\mathcal{M}^2}\mathrm{d}t\Big] < \infty \Big\}.$$

Next, we focus on the maximum principle of minimizing (3.19) subject to (2.8) and (2.10). For this purpose, define the Hamiltonian function $H : [0,T]\times\mathbb{R}^L\times U\times S\times\mathbb{R}^L\times\mathbb{R}^{L\times N}\times\mathbb{R}^{L\times K}\times\mathcal{L}^2(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0),\nu_\alpha;\mathbb{R}^{L\times M})\times\mathcal{M}^2(\mathbb{R}^+;\mathbb{R}^{L\times D})\times\mathbb{R}^K\to\mathbb{R}$ as follows:

$$\begin{aligned} &H(t,x,u,e_i,p,q,k,r,s,z)\\ &:= \langle p, b(t,x,u,e_i)\rangle + \sum_{j=1}^N \langle q_j,\sigma^j(t,x,u,e_i)\rangle + \sum_{j=1}^K\langle k_j,\beta^j(t,x,u,e_i)\rangle\\ &\quad + \sum_{j=1}^M\int_{\mathbb{R}_0}\langle r_j(t,\zeta),\eta^j(t,x,u,e_i,\zeta)\rangle\nu^j_{e_i}(\mathrm{d}\zeta)\\ &\quad + \sum_{j=1}^D\langle s_j,\gamma^j(t,x,u,e_i)\rangle\lambda_{ij} + \langle z, h(t,x,u,e_i)\rangle. \end{aligned} \tag{3.22}$$

We now introduce the adjoint equations which depend on the risk-sensitive parameter θ,

$$\begin{cases} \mathrm{d}a(t) = c(t)\mathrm{d}W(t) + z(t)\mathrm{d}B(t),\\ a(T) = e^{\theta g(\bar{x}(T),\alpha(T))}, \end{cases} \tag{3.23}$$

$$\begin{cases} \mathrm{d}p(t) = -H_x(t,\bar{x}(t),\bar{u}(t),\alpha(t),p(t),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t)^\top\beta(t))\mathrm{d}t\\ \qquad\qquad + \sum\limits_{j=1}^N q_j(t)\mathrm{d}W_j(t) + \sum\limits_{j=1}^K k_j(t)\mathrm{d}B_j(t)\\ \qquad\qquad + \sum\limits_{j=1}^M\int_{\mathbb{R}_0} r_j(t,\zeta)\widetilde{N}^j_\alpha(\mathrm{d}t,\mathrm{d}\zeta) + \sum\limits_{j=1}^D s_j(t)\mathrm{d}\widetilde{\Phi}_j(t),\\ p(T) = \theta e^{\theta g(\bar{x}(T),\alpha(T))}g_x(\bar{x}(T),\alpha(T)), \end{cases} \tag{3.24}$$

$$\begin{cases} \mathrm{d}P(t) = -\Big\{ b_x(t)^\top P(t) + P(t)b_x(t) + \sum\limits_{j=1}^N\big[\sigma^j_x(t)^\top P(t)\sigma^j_x(t) + \sigma^j_x(t)^\top Q_j(t) + Q_j(t)\sigma^j_x(t)\big]\\ \qquad\qquad + \sum\limits_{j=1}^K\big[\beta^j_x(t)^\top P(t)\beta^j_x(t) + \beta^j_x(t)^\top V_j(t) + V_j(t)\beta^j_x(t)\big]\\ \qquad\qquad + \sum\limits_{j=1}^M\int_{\mathbb{R}_0}\big[\eta^j_x(t,\zeta)^\top P(t)\eta^j_x(t,\zeta) + \eta^j_x(t,\zeta)^\top R_j(t,\zeta)\eta^j_x(t,\zeta)\\ \qquad\qquad\qquad + \eta^j_x(t,\zeta)^\top R_j(t,\zeta) + R_j(t,\zeta)\eta^j_x(t,\zeta)\big]\nu^j_\alpha(\mathrm{d}\zeta)\\ \qquad\qquad + \sum\limits_{j=1}^D\big[\gamma^j_x(t)^\top P(t)\gamma^j_x(t) + \gamma^j_x(t)^\top S_j(t)\gamma^j_x(t) + \gamma^j_x(t)^\top S_j(t) + S_j(t)\gamma^j_x(t)\big]\lambda_j(t)\\ \qquad\qquad + H_{xx}(t,\bar{x}(t),\bar{u}(t),\alpha(t),p(t),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t)^\top\beta(t))\\ \qquad\qquad + (k(t)-P(t)\beta(t))h_x(t) + h_x(t)^\top(k(t)^\top-\beta(t)^\top P(t))\Big\}\mathrm{d}t\\ \qquad\qquad + \sum\limits_{j=1}^N Q_j(t)\mathrm{d}W_j(t) + \sum\limits_{j=1}^K V_j(t)\mathrm{d}B_j(t) + \sum\limits_{j=1}^M\int_{\mathbb{R}_0}R_j(t,\zeta)\widetilde{N}^j_\alpha(\mathrm{d}t,\mathrm{d}\zeta) + \sum\limits_{j=1}^D S_j(t)\mathrm{d}\widetilde{\Phi}_j(t),\\ P(T) = \theta^2 e^{\theta g(\bar{x}(T),\alpha(T))}g_x(\bar{x}(T),\alpha(T))g_x^\top(\bar{x}(T),\alpha(T)) + \theta e^{\theta g(\bar{x}(T),\alpha(T))}g_{xx}(\bar{x}(T),\alpha(T)), \end{cases} \tag{3.25}$$

where $\beta(t) = \beta(t,\bar{x}(t),\bar{u}(t),\alpha(t))$. Here (3.23) is used to treat the term Z raised by partial information and (3.24)-(3.25) are similar to the full information case. Obviously, Assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$ imply that (3.23)-(3.25) admit unique solutions.

Then we have the following theorem.

Theorem 3.4. *(Risk-Sensitive Maximum Principle: I) Let Assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$ hold and $\bar{u}(\cdot)$ be an optimal control. Then there exist unique solutions $(a(\cdot),c(\cdot),z(\cdot))\in L^2_{\mathcal{F}}([0,T];\mathbb{R})\times L^2_{\mathcal{F}}([0,T];\mathbb{R}^N)\times L^2_{\mathcal{F}}([0,T];\mathbb{R}^K)$, $(p(\cdot),q(\cdot),k(\cdot),r(\cdot,\cdot),s(\cdot))\in L^2_{\mathcal{F}}([0,T];\mathbb{R}^L)\times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^{L\times N})\times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^{L\times K})\times F^2_p([0,T];\mathbb{R}^{L\times M})\times M^2_p([0,T];\mathbb{R}^{L\times D})$ and $(P(\cdot),Q(\cdot),V(\cdot),R(\cdot,\cdot),S(\cdot))\in L^2_{\mathcal{F}}([0,T];\mathbb{R}^{L\times L})\times\big(L^2_{\mathcal{F},p}([0,T];\mathbb{R}^{L\times L})\big)^N\times\big(L^2_{\mathcal{F},p}([0,T];\mathbb{R}^{L\times L})\big)^K\times\big(F^2_p([0,T];\mathbb{R}^{L\times L})\big)^M\times$*

$\big(M_p^2([0,T];\mathbb{R}^{L\times L})\big)^D$ *to* (3.23), (3.24) *and* (3.25) *respectively, such that*

$$\begin{aligned}
E^{\bar u}\Big\{&H(t,\bar x(t-),u,\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&-H(t,\bar x(t-),\bar u(t),\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&+\frac12\mathrm{tr}\Big[P(t-)(\delta\sigma(t,u)\delta\sigma(t,u)^\top+\delta\beta(t,u)\delta\beta(t,u)^\top)\Big]\\
&+\frac12\sum_{j=1}^M\mathrm{tr}\Big[\int_{\mathbb{R}_0}(P(t-)+R_j(t,\zeta))\delta\eta^j(t,u,\zeta)\delta\eta^j(t,u,\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]\\
&+\frac12\sum_{j=1}^D\mathrm{tr}\Big[(P(t-)+S_j(t))\delta\gamma^j(t,u)\delta\gamma^j(t,u)^\top\lambda_j(t)\Big]\Big|\mathcal{F}_t^Y\Big\}\geq0,\\
&\qquad\forall u\in U,\quad \text{a.e. } t\in[0,T],\ \ P\text{-a.s.}
\end{aligned}\tag{3.26}$$

Proof. Applying Itô's formula ([6, Theorem 4.1]) to

$$t\to\langle a(t),\bar Z^{-1}(t)(Z_1(t)+Z_2(t))\rangle+\langle p(t),\bar Z^{-1}(t)Z_1(t)x_1(t)\rangle,$$

we have

$$\begin{aligned}
&E^{\bar u}\Big[\bar Z^{-1}(T)(Z_1(T)+Z_2(T))e^{\theta g(\bar x(T),\alpha(T))}\Big]+\theta E^{\bar u}\Big[\bar Z^{-1}(T)Z_1(T)e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))x_1(T)\Big]\\
&=E^{\bar u}\int_0^T\sum_{j=1}^K\langle p(t-),x_1(t)^\top h_x^j(t)^\top\beta_x^j(t)x_1(t)+\delta h^j(t,u(t))\delta\beta^j(t,u(t))I_{E_\varepsilon}(t)\rangle\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\langle z(t),h_x(t)(x_1(t)+x_2(t))+\delta h(t,u(t))I_{E_\varepsilon}(t)\rangle\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\sum_{j=1}^K x_1(t)^\top k_j(t)h_x^j(t)x_1(t)\mathrm{d}t+\frac12E^{\bar u}\int_0^T\langle z(t),h_{xx}(t)x_1(t)^2\rangle\mathrm{d}t+o(\varepsilon),
\end{aligned}\tag{3.27}$$

where $h(t)=(h^1(t),\dots,h^K(t))^\top$ and $h_x^j(t)=(\frac{\partial h^j}{\partial x^1}(t),\dots,\frac{\partial h^j}{\partial x^L}(t))\triangleq(h^j_{x^1}(t),\dots,h^j_{x^L}(t))$.

Applying Itô's formula to $t\to\langle p(t),x_1(t)+x_2(t)\rangle$, we get

$$\begin{aligned}
&E^{\bar u}\Big[\theta e^{\theta g(\bar x(T),\alpha(T))}g_x^\top(\bar x(T),\alpha(T))(x_1(T)+x_2(T))\Big]\\
&=E^{\bar u}\int_0^T\Big[\langle p(t-),\delta\hat b(t,u(t))I_{E_\varepsilon}(t)+\frac12\hat b_{xx}(t)x_1(t)^2\rangle-\langle z(t),h_x(t)(x_1(t)+x_2(t))\rangle\Big]\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\sum_{j=1}^K\langle p(t-),\delta\beta^j(t,u(t))h^j(t)I_{E_\varepsilon}(t)+\frac12\beta^j_{xx}(t)h^j(t)x_1(t)^2\rangle\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\sum_{j=1}^N\langle q_j(t),\delta\sigma^j(t,u(t))I_{E_\varepsilon}(t)+\frac12\sigma^j_{xx}(t)x_1(t)^2\rangle\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\sum_{j=1}^K\langle k_j(t),\delta\beta^j(t,u(t))I_{E_\varepsilon}(t)+\frac12\beta^j_{xx}(t)x_1(t)^2\rangle\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\int_{\mathbb{R}_0}\sum_{j=1}^M\langle r_j(t,\zeta),\delta\eta^j(t,u(t),\zeta)I_{E_\varepsilon}(t)+\frac12\eta^j_{xx}(t,\zeta)x_1(t-)^2\rangle\nu^j_\alpha(\mathrm{d}\zeta)\mathrm{d}t\\
&\quad+E^{\bar u}\int_0^T\sum_{j=1}^D\langle s_j(t),\delta\gamma^j(t,u(t))I_{E_\varepsilon}(t)+\frac12\gamma^j_{xx}(t)x_1(t-)^2\rangle\lambda_j(t)\mathrm{d}t+o(\varepsilon).
\end{aligned}\tag{3.28}$$

Applying once more Itô's formula to $t\to\mathrm{tr}\big[P(t)x_1(t)x_1(t)^\top\big]$, we have

$$\frac12\theta^2E^{\bar u}\Big[e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_x(\bar x(T),\alpha(T))g_x^\top(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]$$

$$
\begin{aligned}
&+\frac{1}{2}\theta E^{\bar u}\Big[e^{\theta g(\bar x(T),\alpha(T))}\mathrm{tr}\big\{g_{xx}(\bar x(T),\alpha(T))x_1(T)x_1(T)^\top\big\}\Big]\\
&=\frac{1}{2}E^{\bar u}\int_0^T\mathrm{tr}\Big[P(t-)(\delta\sigma(t,u(t))\delta\sigma(t,u(t))^\top+\delta\beta(t,u(t))\delta\beta(t,u(t))^\top)\Big]I_{E_\varepsilon}(t)\mathrm{d}t\\
&+\frac{1}{2}E^{\bar u}\int_0^T\sum_{j=1}^M\mathrm{tr}\Big[\int_{\mathbb{R}_0}(P(t-)+R_j(t,\zeta))\delta\eta^j(t,u(t),\zeta)\delta\eta^j(t,u(t),\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]I_{E_\varepsilon}(t)\mathrm{d}t\\
&+\frac{1}{2}E^{\bar u}\int_0^T\sum_{j=1}^D\mathrm{tr}\Big[(P(t-)+S_j(t))\delta\gamma^j(t,u(t))\delta\gamma^j(t,u(t))^\top\lambda_j(t)\Big]I_{E_\varepsilon}(t)\mathrm{d}t\\
&-\frac{1}{2}E^{\bar u}\int_0^T\mathrm{tr}\Big[H_{xx}(t,\bar x(t-),\bar u(t),\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),\\
&\qquad\qquad z(t)-p(t-)^\top\beta(t))x_1(t-)x_1(t-)^\top\Big]\mathrm{d}t\\
&-E^{\bar u}\int_0^T\sum_{j=1}^K x_1(t)^\top k_j(t)h_x^j(t)x_1(t)\mathrm{d}t+o(\varepsilon).
\end{aligned}\tag{3.29}
$$

Substituting (3.27), (3.28) and (3.29) into (3.20), we obtain

$$
\begin{aligned}
E^{\bar u}\int_0^T&\Big\{H(t,\bar x(t-),u(t),\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&-H(t,\bar x(t-),\bar u(t),\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&+\frac{1}{2}\mathrm{tr}\Big[P(t-)(\delta\sigma(t,u(t))\delta\sigma(t,u(t))^\top+\delta\beta(t,u(t))\delta\beta(t,u(t))^\top)\Big]\\
&+\frac{1}{2}\sum_{j=1}^M\mathrm{tr}\Big[\int_{\mathbb{R}_0}(P(t-)+R_j(t,\zeta))\delta\eta^j(t,u(t),\zeta)\delta\eta^j(t,u(t),\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]\\
&+\frac{1}{2}\sum_{j=1}^D\mathrm{tr}\Big[(P(t-)+S_j(t))\delta\gamma^j(t,u(t))\delta\gamma^j(t,u(t))^\top\lambda_j(t)\Big]\Big\}I_{E_\varepsilon}(t)\mathrm{d}t\geq o(\varepsilon).
\end{aligned}\tag{3.30}
$$

Let $u\in U$ and A be an arbitrary element of $\mathcal{F}_t^Y$, set $u(\cdot)=uI_A(\cdot)+\bar u(\cdot)I_{A^C}(\cdot)$. Dividing (3.30) by ε, letting ε go to 0 and taking $u(\cdot)$ into the inequality yield

$$
\begin{aligned}
E^{\bar u}\Big\{\Big[&H(t,\bar x(t-),u,\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&-H(t,\bar x(t-),\bar u(t),\alpha(t-),p(t-),q(t),k(t),r(t,\cdot),s(t),z(t)-p(t-)^\top\beta(t))\\
&+\frac{1}{2}\mathrm{tr}\Big[P(t-)(\delta\sigma(t,u)\delta\sigma(t,u)^\top+\delta\beta(t,u)\delta\beta(t,u)^\top)\Big]\\
&+\frac{1}{2}\sum_{j=1}^M\mathrm{tr}\Big[\int_{\mathbb{R}_0}(P(t-)+R_j(t,\zeta))\delta\eta^j(t,u,\zeta)\delta\eta^j(t,u,\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]\\
&+\frac{1}{2}\sum_{j=1}^D\mathrm{tr}\Big[(P(t-)+S_j(t))\delta\gamma^j(t,u)\delta\gamma^j(t,u)^\top\lambda_j(t)\Big]\Big]I_A\Big\}\geq 0,\ \text{a.e. } t\in[0,T],
\end{aligned}\tag{3.31}
$$

which implies the variational inequality (3.26) holds. □

3.2. **The case of "$f\neq 0$".**

In this subsection, we study the problem with a general running cost functional, that is,

$$
\begin{aligned}
&\min_{u(\cdot)\in\mathcal{U}[0,T]}J(x_0,e_i;u(\cdot)),\\
&J(x_0,e_i;u(\cdot)):=E\Big[Z(T)e^{\theta\big(g(x(T),\alpha(T))+\int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\big)}\Big],
\end{aligned}\tag{3.32}
$$

subject to (2.8) and (2.10). The objective is to derive a general necessary condition for the optimal control $\bar{u}(\cdot)$. To this end, we assume throughout this subsection that $L = 1$, that is, the control system (2.8) is of 1-dimensional. The method is to combine the proof of Theorem 3.4 with a reformulation of the cost functional (3.32).

Define the following stochastic differential equation:

$$\mathrm{d}y(t) = f(t, x(t), u(t), \alpha(t))\mathrm{d}t, \qquad y(0) = 0, \tag{3.33}$$

and let $\bar{y}(\cdot)$ be the solution to (3.33) under the control $\bar{u}(\cdot)$. Similar to (3.3), the first-order variational equation for (3.33) is given by

$$\mathrm{d}y_1(t) = \Big\{ f_x(t)y_1(t) + \delta f(t, u(t))I_{E_\varepsilon}(t) \Big\}\mathrm{d}t, \qquad y_1(0) = 0. \tag{3.34}$$

For any $u(\cdot) \in \mathcal{U}[0, T]$ and $k \geq 1$, we can employ the usual techniques to prove that

$$\sup_{t\in[0,T]} E|y_1(t)|^{2k} = O(\varepsilon^{2k}), \tag{3.35}$$

$$\sup_{t\in[0,T]} E|y^\varepsilon(t) - \bar{y}(t) - y_1(t)|^{2k} = o(\varepsilon^{2k}). \tag{3.36}$$

Therefore, our original problem (3.32), subject to (2.8) and (2.10), is equivalent to minimizing

$$J(x_0, e_i; u(\cdot)) := E\Big[Z(T)e^{\theta\big(g(x(T),\alpha(T))+y(T)\big)} \Big], \tag{3.37}$$

subject to (2.8), (2.10) and (3.33). The fact that $J(x_0, e_i; u^\varepsilon(\cdot)) - J(x_0, e_i; \bar{u}(\cdot)) \geq 0$ implies that

$$\begin{aligned}
&E^{\bar{u}}\Big[\bar{Z}^{-1}(T)(Z_1(T) + Z_2(T))e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} \Big] \\
&\quad +\theta E^{\bar{u}}\Big[\bar{Z}^{-1}(T)Z_1(T)e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} g_x^\top(\bar{x}(T), \alpha(T))x_1(T) \Big] \\
&\quad +\theta E^{\bar{u}}\Big[e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} g_x^\top(\bar{x}(T), \alpha(T))(x_1(T) + x_2(T)) \Big] \\
&\quad +\theta E^{\bar{u}}\Big[e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} y_1(T) \Big] \\
&\quad +\frac{1}{2}\theta^2 E^{\bar{u}}\Big[e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} \mathrm{tr}\big\{ g_x(\bar{x}(T), \alpha(T))g_x^\top(\bar{x}(T), \alpha(T))x_1(T)x_1(T)^\top \big\} \Big] \\
&\quad +\frac{1}{2}\theta E^{\bar{u}}\Big[e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} \mathrm{tr}\big\{ g_{xx}(\bar{x}(T), \alpha(T))x_1(T)x_1(T)^\top \big\} \Big] \geq o(\varepsilon).
\end{aligned} \tag{3.38}$$

It is clear that (3.38) is similar to (3.20). As in the previous subsection, we introduce the following adjoint equations which are four finite-dimensional BSDEs:

$$\begin{cases} \mathrm{d}\tilde{\xi}(t) = -f_x(t)\tilde{\xi}(t)\mathrm{d}t + \tilde{\pi}(t)\mathrm{d}W(t), \\ \tilde{\xi}(T) = \theta e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)}, \end{cases} \tag{3.39}$$

$$\begin{cases} \mathrm{d}\tilde{a}(t) = \tilde{c}(t)\mathrm{d}W(t) + \tilde{z}(t)\mathrm{d}B(t), \\ \tilde{a}(T) = e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)}, \end{cases} \tag{3.40}$$

$$\begin{cases} \mathrm{d}\tilde{p}(t) = -H_x(t, \bar{x}(t), \bar{u}(t), \alpha(t), \tilde{p}(t), \tilde{q}(t), \tilde{k}(t), \tilde{r}(t,\cdot), \tilde{s}(t), \tilde{z}(t) - \tilde{p}(t)^\top\beta(t))\mathrm{d}t \\ \qquad\qquad + \sum_{j=1}^{N} \tilde{q}_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^{K} \tilde{k}_j(t)\mathrm{d}B_j(t) \\ \qquad\qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{r}_j(t,\zeta)\widetilde{N}_\alpha^j(\mathrm{d}t, \mathrm{d}\zeta) + \sum_{j=1}^{D} \tilde{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ \tilde{p}(T) = \theta e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)} g_x(\bar{x}(T), \alpha(T)), \end{cases} \tag{3.41}$$

$$\begin{cases} \mathrm{d}\tilde{P}(t) = -\Big\{ b_x(t)^\top \tilde{P}(t) + \tilde{P}(t) b_x(t) \\ \qquad + \sum_{j=1}^{N} \big[\sigma_x^j(t)^\top \tilde{P}(t)\sigma_x^j(t) + \sigma_x^j(t)^\top \tilde{Q}_j(t) + \tilde{Q}_j(t)\sigma_x^j(t)\big] \\ \qquad + \sum_{j=1}^{K} \big[\beta_x^j(t)^\top \tilde{P}(t)\beta_x^j(t) + \beta_x^j(t)^\top \tilde{V}_j(t) + \tilde{V}_j(t)\beta_x^j(t)\big] \\ \qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \big[\eta_x^j(t,\zeta)^\top \tilde{P}(t)\eta_x^j(t,\zeta) + \eta_x^j(t,\zeta)^\top \tilde{R}_j(t,\zeta)\eta_x^j(t,\zeta) \\ \qquad\qquad + \eta_x^j(t,\zeta)^\top \tilde{R}_j(t,\zeta) + \tilde{R}_j(t,\zeta)\eta_x^j(t,\zeta)\big]\nu_\alpha^j(\mathrm{d}\zeta) \\ \qquad + \sum_{j=1}^{D} \big[\gamma_x^j(t)^\top \tilde{P}(t)\gamma_x^j(t) + \gamma_x^j(t)^\top \tilde{S}_j(t)\gamma_x^j(t) \\ \qquad\qquad + \gamma_x^j(t)^\top \tilde{S}_j(t) + \tilde{S}_j(t)\gamma_x^j(t)\big]\lambda_j(t) \\ \qquad + H_{xx}(t,\bar{x}(t),\bar{u}(t),\alpha(t),\tilde{p}(t),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t)^\top\beta(t)) \\ \qquad + (\tilde{k}(t) - \tilde{P}(t)\beta(t))h_x(t) + h_x(t)^\top(\tilde{k}(t)^\top - \beta(t)^\top\tilde{P}(t))\Big\}\mathrm{d}t \\ \qquad + \sum_{j=1}^{N} \tilde{Q}_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^{K} \tilde{V}_j(t)\mathrm{d}B_j(t) \\ \qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{R}_j(t,\zeta)\tilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}\zeta) + \sum_{j=1}^{D} \tilde{S}_j(t)\mathrm{d}\tilde{\Phi}_j(t), \\ \tilde{P}(T) = \theta^2 e^{\theta\left(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\right)} g_x(\bar{x}(T),\alpha(T)g_x^\top(\bar{x}(T),\alpha(T)) \\ \qquad + \theta e^{\theta\left(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\right)} g_{xx}(\bar{x}(T),\alpha(T)). \end{cases} \tag{3.42}$$

Here (3.39) is used to treat the additional state variable y and (3.40)-(3.42) are similar to (3.23)-(3.25), but with different terminal conditions. Under Assumptions ($\mathcal{A}$1)-($\mathcal{A}$3), there exist unique solutions $(\tilde{\xi}(\cdot),\tilde{\pi}(\cdot)) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}) \times L^2_{\mathcal{F}}([0,T];\mathbb{R}^N)$; $(\tilde{a}(\cdot),\tilde{c}(\cdot),\tilde{z}(\cdot)) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}) \times L^2_{\mathcal{F}}([0,T];\mathbb{R}^N) \times L^2_{\mathcal{F}}([0,T];\mathbb{R}^K)$; $(\tilde{p}(\cdot),\tilde{q}(\cdot),\tilde{k}(\cdot),\tilde{r}(\cdot,\cdot),\tilde{s}(\cdot)) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}) \times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^N) \times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^K) \times F^2_p([0,T];\mathbb{R}^M) \times M^2_p([0,T];\mathbb{R}^D)$; $(\tilde{P}(\cdot),\tilde{Q}(\cdot),\tilde{V}(\cdot),\tilde{R}(\cdot,\cdot),\tilde{S}(\cdot)) \in L^2_{\mathcal{F}}([0,T];\mathbb{R}) \times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^N) \times L^2_{\mathcal{F},p}([0,T];\mathbb{R}^K) \times F^2_p([0,T];\mathbb{R}^M) \times M^2_p([0,T];\mathbb{R}^D)$ satisfying (3.39)-(3.42).

Define a new Hamiltonian function $\widetilde{H} : [0,T]\times\mathbb{R}\times U\times S\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^K\times\mathcal{L}^2(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0),\nu_\alpha;\mathbb{R}^M)\times\mathcal{M}^2(\mathbb{R}^+;\mathbb{R}^D)\times\mathbb{R}^K\times\mathbb{R}\to\mathbb{R}$ by

$$\widetilde{H}(t,x,u,e_i,\tilde{p},\tilde{q},\tilde{k},\tilde{r},\tilde{s},\tilde{z},\tilde{\xi}) = H(t,x,u,e_i,\tilde{p},\tilde{q},\tilde{k},\tilde{r},\tilde{s},\tilde{z}) + \langle\tilde{\xi}, f(t,x,u,e_i)\rangle. \tag{3.43}$$

Then applying Itô's formula to $t \to \langle\tilde{\xi}(t), y_1(t)\rangle$, we have

$$\theta E^{\bar{u}}\Big[e^{\theta\left(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\right)} y_1(T)\Big] = E^{\bar{u}}\int_0^T \langle\tilde{\xi}(t), \delta f(t,u(t)) I_{E_\varepsilon}(t)\rangle\mathrm{d}t. \tag{3.44}$$

From (3.38), (3.43), (3.44) and Theorem 3.4, we get

$$\begin{aligned} E^{\bar{u}}\Big\{ & \widetilde{H}(t,\bar{x}(t-),u,\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)) \\ & -\widetilde{H}(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)) \\ & +\frac{1}{2}\mathrm{tr}\Big[\tilde{P}(t-)(\delta\sigma(t,u)\delta\sigma(t,u)^\top + \delta\beta(t,u)\delta\beta(t,u)^\top)\Big] \\ & +\frac{1}{2}\sum_{j=1}^{M}\mathrm{tr}\Big[\int_{\mathbb{R}_0}(\tilde{P}(t-)+\tilde{R}_j(t,\zeta))\delta\eta^j(t,u,\zeta)\delta\eta^j(t,u,\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big] \\ & +\frac{1}{2}\sum_{j=1}^{D}\mathrm{tr}\Big[(\tilde{P}(t-)+\tilde{S}_j(t))\delta\gamma^j(t,u)\delta\gamma^j(t,u)^\top\lambda_j(t)\Big]\Big|\mathcal{F}_t^Y\Big\} \geq 0, \\ & \forall u\in U, \quad \text{a.e. } t\in[0,T], \ \text{P-a.s.} \end{aligned} \tag{3.45}$$

Consequently, the above procedure yields the following theorem.

Theorem 3.5. *(Risk-Sensitive Maximum Principle: II) Let $L=1$ and Assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$ hold. Suppose that $\bar{u}(\cdot)$ is an optimal control for problem (3.32). Then the variational inequality (3.45) holds. Or equivalently,*

$$E^{\bar{u}}\Big\{\widetilde{\mathcal{H}}(t,\bar{x}(t-),u,\alpha(t-))-\widetilde{\mathcal{H}}(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\Big|\mathcal{F}_t^Y\Big\}\geq 0,\ \forall u\in U,\ \text{a.e. } t\in[0,T],\ P\text{-a.s.},$$

where $\widetilde{\mathcal{H}}:[0,T]\times\mathbb{R}\times U\times S\to\mathbb{R}$ is defined by:

$$\begin{aligned}
&\widetilde{\mathcal{H}}(t,x,u,e_i)=\widetilde{H}(t,x,u,e_i,\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\\
&\quad-\frac{1}{2}\mathrm{tr}\Big[\tilde{P}(t-)\Big(\sigma(t,\bar{x}(t),\bar{u}(t),e_i)\sigma(t,\bar{x}(t),\bar{u}(t),e_i)^\top+\beta(t,\bar{x}(t),\bar{u}(t),e_i)\beta(t,\bar{x}(t),\bar{u}(t),e_i)^\top\Big)\Big]\\
&\quad+\frac{1}{2}\mathrm{tr}\Big[\tilde{P}(t-)\Big(\Delta\sigma(t,u)\Delta\sigma(t,u)^\top+\Delta\beta(t,u)\Delta\beta(t,u)^\top\Big)\Big]\\
&\quad-\frac{1}{2}\sum_{j=1}^{M}\mathrm{tr}\Big[\int_{\mathbb{R}_0}\Big(\tilde{P}(t-)+\tilde{R}_j(t,\zeta)\Big)\eta^j(t,\bar{x}(t),\bar{u}(t),e_i,\zeta)\eta^j(t,\bar{x}(t),\bar{u}(t),e_i,\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]\\
&\quad+\frac{1}{2}\sum_{j=1}^{M}\mathrm{tr}\Big[\int_{\mathbb{R}_0}\Big(\tilde{P}(t-)+\tilde{R}_j(t,\zeta)\Big)\Delta\eta^j(t,u,\zeta)\Delta\eta^j(t,u,\zeta)^\top\nu_\alpha^j(\mathrm{d}\zeta)\Big]\\
&\quad-\frac{1}{2}\sum_{j=1}^{D}\mathrm{tr}\Big[\Big(\tilde{P}(t-)+\tilde{S}_j(t)\Big)\gamma^j(t,\bar{x}(t),\bar{u}(t),e_i)\gamma^j(t,\bar{x}(t),\bar{u}(t),e_i)^\top\lambda_{ij}\Big]\\
&\quad+\frac{1}{2}\sum_{j=1}^{D}\mathrm{tr}\Big[\Big(\tilde{P}(t-)+\tilde{S}_j(t)\Big)\Delta\gamma^j(t,u)\Delta\gamma^j(t,u)^\top\lambda_{ij}\Big],
\end{aligned}$$

with $\Delta\sigma(t,u):=\sigma(t,x,u,e_i)-\sigma(t,\bar{x}(t-),\bar{u}(t),e_i)$ and similarly for β,η^j,γ^j.

Remark 3.6. *Following from (3.34) and (3.39), we can see that only when $L=1$ will these two equations make sense.*

3.3. **Sufficient condition for the optimality.**

In this subsection, we develop a sufficient condition for the optimality of problem (2.14) under some convexity assumptions. It is worth mentioning that our approach used here is different from that of [18]. In addition to Assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$, we also need the following hypotheses.

$(\mathcal{A}4)$ The control domain U is convex.

$(\mathcal{A}5)$ For any $(x,u)\in\mathbb{R}\times U$, all the coefficients including $b,\sigma,\beta,\eta,\gamma,h,f$ are differentiable in u and f does not depend on x.

Theorem 3.7. *(Sufficient Condition for the Optimality) Let Assumptions $(\mathcal{A}1)$-$(\mathcal{A}5)$ hold and $(\bar{u}(\cdot),\bar{x}(\cdot),\bar{y}(\cdot))$ be a candidate optimal triple. Suppose that $(\tilde{\xi}(\cdot),\tilde{\pi}(\cdot))$, $(\tilde{a}(\cdot),\tilde{c}(\cdot),\tilde{z}(\cdot))$ and $(\tilde{p}(\cdot),\tilde{q}(\cdot),\tilde{k}(\cdot),\tilde{r}(\cdot,\cdot),\tilde{s}(\cdot))$ satisfy (3.39), (3.40) and (3.41) respectively such that, for any $u(\cdot)\in\mathcal{U}[0,T]$, we have*

$$E^u\int_0^T|\widetilde{H}_u(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))|^2\mathrm{d}t<+\infty.$$

Moreover, suppose that for all $(t,x,u,e_i)\in[0,T]\times\mathbb{R}^L\times U\times S$, $Z(t)$ is $\mathcal{F}_t^Y$-adapted, the Hamiltonian $\widetilde{H}$ is convex in (x,u) and g is convex in x, and

$$\begin{aligned}
&E\Big[\widetilde{H}(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\big|\mathcal{F}_t^Y\Big]\\
&=\min_{u\in U}E\Big[\widetilde{H}(t,\bar{x}(t-),u,\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\big|\mathcal{F}_t^Y\Big].\quad(3.46)
\end{aligned}$$

Then $\bar{u}(\cdot)$ is an optimal control.

Proof. For any $u(\cdot) \in \mathcal{U}[0,T]$, by virtue of the convexity property of e^x and $g(x,e_i)$, we have

$$J(x_0,e_i;u(\cdot)) - J(x_0,e_i;\bar{u}(\cdot)) \geq I_1 + I_2 + I_3, \tag{3.47}$$

with

$$\begin{aligned}
I_1 &= E\Big[\big(Z(T)-\bar{Z}(T)\big)e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)}\Big],\\
I_2 &= E^u\Big[\big(y(T)-\bar{y}(T)\big)\theta e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)}\Big],\\
I_3 &= E^u\Big[\big(x(T)-\bar{x}(T)\big)\theta e^{\theta\big(g(\bar{x}(T),\alpha(T))+\bar{y}(T)\big)}g_x(\bar{x}(T),\alpha(T))\Big].
\end{aligned}$$

Applying Itô's formula to

$$t \to \langle Z(t)-\bar{Z}(t), \tilde{a}(t)\rangle,\ t \to \langle y(t)-\bar{y}(t), \tilde{\xi}(t)\rangle \text{ and } t \to \langle x(t)-\bar{x}(t), \tilde{p}(t)\rangle,$$

we have

$$I_1 = E^u\int_0^T \langle \tilde{z}(t), h(t,x(t-),u(t),\alpha(t-)) - h(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t, \tag{3.48}$$

$$I_2 = E^u\int_0^T \langle \tilde{\xi}(t), f(t,u(t),\alpha(t-)) - f(t,\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t, \tag{3.49}$$

$$\begin{aligned}
I_3 = {} & E^u\int_0^T \langle \tilde{p}(t-), b(t,x(t-),u(t),\alpha(t-)) - b(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t\\
& - E^u\int_0^T \langle \tilde{p}(t-)^T\beta(t), h(t,x(t-),u(t),\alpha(t-)) - h(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t\\
& + E^u\int_0^T \langle \tilde{q}(t), \sigma(t,x(t-),u(t),\alpha(t-)) - \sigma(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t\\
& + E^u\int_0^T \langle \tilde{k}(t), \beta(t,x(t-),u(t),\alpha(t-)) - \beta(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \mathrm{d}t\\
& + E^u\int_0^T \sum_{j=1}^M \int_{\mathbb{R}_0} \langle \tilde{r}_j(t,\zeta), \eta^j(t,x(t-),u(t),\alpha(t-),\zeta) - \eta^j(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\zeta)\rangle \nu_\alpha^j(\mathrm{d}\zeta)\mathrm{d}t\\
& + E^u\int_0^T \sum_{j=1}^D \langle \tilde{s}_j(t), \gamma^j(t,x(t-),u(t),\alpha(t-)) - \gamma^j(t,\bar{x}(t-),\bar{u}(t),\alpha(t-))\rangle \lambda_j(t)\mathrm{d}t\\
& - E^u\int_0^T \langle H_x(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t)),\\
& \qquad x(t-)-\bar{x}(t-)\rangle \mathrm{d}t.
\end{aligned} \tag{3.50}$$

Substituting (3.48), (3.49) and (3.50) into (3.47), noting the definition of $\widetilde{H}$ and its convexity property, we get

$$\begin{aligned}
& J(x_0,e_i;u(\cdot)) - J(x_0,e_i;\bar{u}(\cdot))\\
& \geq E^u\int_0^T \widetilde{H}(t,\bar{x}(t-),u(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\mathrm{d}t\\
& \quad - E^u\int_0^T \widetilde{H}(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\mathrm{d}t\\
& \quad - E^u\int_0^T \langle \widetilde{H}_x(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)),\\
& \qquad x(t-)-\bar{x}(t-)\rangle \mathrm{d}t\\
& \geq E^u\int_0^T \langle \widetilde{H}_u(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)),\\
& \qquad u(t)-\bar{u}(t)\rangle \mathrm{d}t
\end{aligned}$$

$$= E\int_0^T Z(t)E\Big[\langle \widetilde{H}_u(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)), \\ u(t)-\bar{u}(t)\rangle|\mathcal{F}_t^Y\Big]\mathrm{d}t. \tag{3.51}$$

Since $u=\bar{u}(t)$ minimizes

$$u\to E\Big[\widetilde{H}(t,\bar{x}(t-),u,\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi})|\mathcal{F}_t^Y\Big], \tag{3.52}$$

we deduce that

$$\frac{d}{du}E\Big[\widetilde{H}(t,\bar{x}(t-),u,\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot), \\ \tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi})|\mathcal{F}_t^Y\Big]_{u=\bar{u}(t)}(u(t)-\bar{u}(t))\geq 0, \tag{3.53}$$

i.e.,

$$E\Big[\langle\widetilde{H}_u(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot), \\ \tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t)),u(t)-\bar{u}(t)\rangle|\mathcal{F}_t^Y\Big]\mathrm{d}t\geq 0. \tag{3.54}$$

Substituting (3.54) into (3.51) and noting $Z(t)>0$, we have

$$J(x_0,e_i;u(\cdot))-J(x_0,e_i;\bar{u}(\cdot))\geq 0, \tag{3.55}$$

i.e., $\bar{u}(\cdot)$ is an optimal control. □

Remark 3.8. *It should be noted that our approach in deriving the risk-sensitive maximum principles (Theorems 3.4, 3.5 and 3.7) can be directly extended to the general cost functional case (equation* (2.11)*) under suitable assumptions on the disutility function* Ψ*.*

4. Application to LQ Risk-Sensitive Control Problem

In this section, we are going to investigate a special case of problem (2.13) when the state equation is linear in both the state and control, whereas the exponent part of the cost functional is quadratic. Such a control problem is called an LQ risk-sensitive optimal control problem. It is worth mentioning that the LQ risk-sensitive problems constitute an extremely important class of risk-sensitive optimal control problems, since many problems with financial applications can be eventually formulated as a LQ risk-sensitive problem. For examples of applications see [13, 14, 20, 33]. More importantly, many nonlinear control problems can be reasonably approximated by the LQ problems.

For simplicity, we still adopt the notations introduced above and suppose all the processes are of 1-dimensional except the Markov chain. For each $e_i\in S$ and $\zeta\in\mathbb{R}_0$, take

$$\begin{aligned}
&b(t,x,u,e_i)=A_1(t,e_i)x+C_1(t,e_i)u,\ \sigma(t,x,u,e_i)=A_2(t,e_i)x+C_2(t,e_i)u,\\
&\beta(t,x,u,e_i)=A_3(t,e_i)x+C_3(t,e_i)u,\ \eta(t,x,u,e_i,\zeta)=A_4(t,e_i,\zeta)x+C_4(t,e_i,\zeta)u,\\
&\gamma^j(t,x,u,e_i)=A_5^j(t,e_i)x+C_5^j(t,e_i)u,\ h(t,x,u,e_i)=F(t,e_i),\\
&f(t,x,u,e_i)=\frac{1}{2}u^2,\ g(x,e_i)=G(t,e_i)x^2,
\end{aligned}$$

where $A_k(\cdot,e_i)$, $C_k(\cdot,e_i)$, $F(\cdot,e_i)$ and $G(\cdot,e_i)$ $(i,j=1,\dots D,k=1,\dots,5)$ are bounded deterministic functions and $G(\cdot,e_i)>0$. We now use the risk-sensitive maximum principle established in the previous section to solve this problem.

Similar to [29] or [30], we write down the Hamiltonian function

$$\begin{aligned}
\widetilde{\mathcal{H}}(t,\bar{x}(t-),u,\alpha(t-))=&\Big(A_1(t,\alpha(t-))\bar{x}(t-)+C_1(t,\alpha(t-))u\Big)\tilde{p}(t-)\\
&+\Big(A_2(t,\alpha(t-))\bar{x}(t-)+C_2(t,\alpha(t-))u\Big)\Big[\tilde{q}(t)-\tilde{P}(t-)\big(A_2(t,\alpha(t-))\bar{x}(t-)\\
&+C_2(t,\alpha(t-))\bar{u}(t)\big)\Big]+\frac{1}{2}\tilde{P}(t-)\Big(A_2(t,\alpha(t-))\bar{x}(t-)+C_2(t,\alpha(t-))u\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(A_3(t,\alpha(t-))\bar{x}(t-)+C_3(t,\alpha(t-))u\Big)\Big[\tilde{k}(t)-\tilde{P}(t-)\big(A_3(t,\alpha(t-))\bar{x}(t-)\\
&\quad +C_3(t,\alpha(t-))\bar{u}(t)\big)\Big]+\frac{1}{2}\tilde{P}(t-)\Big(A_3(t,\alpha(t-))\bar{x}(t-)+C_3(t,\alpha(t-))u\Big)^2\\
&+\int_{\mathbb{R}_0}\Big[\Big(A_4(t,\alpha(t-),\zeta)\bar{x}(t-)+C_4(t,\alpha(t-),\zeta)u\Big)\Big[\tilde{r}(t,\zeta)-\big(\tilde{P}(t-)+\tilde{R}(t,\zeta)\big)\\
&\quad\times\big(A_4(t,\alpha(t-),\zeta)\bar{x}(t-)+C_4(t,\alpha(t-),\zeta)\bar{u}(t)\big)\Big]\\
&\quad+\frac{1}{2}\big(\tilde{P}(t-)+\tilde{R}(t,\zeta)\big)\Big(A_4(t,\alpha(t-),\zeta)\bar{x}(t-)+C_4(t,\alpha(t-),\zeta)u\Big)^2\Big]\nu_\alpha(\mathrm{d}\zeta)\\
&+\sum_{j=1}^{D}\Big[\Big(A_5^j(t,\alpha(t-))\bar{x}(t-)+C_5^j(t,\alpha(t-))u\Big)\Big[\tilde{s}_j(t)-\big(\tilde{P}(t-)+\tilde{S}_j(t)\big)\\
&\quad\times\big(A_5^j(t,\alpha(t-))\bar{x}(t-)+C_5^j(t,\alpha(t-))\bar{u}(t)\big)\Big]\\
&\quad+\frac{1}{2}\big(\tilde{P}(t-)+\tilde{S}_j(t)\big)\Big(A_5^j(t,\alpha(t-))\bar{x}(t-)+C_5^j(t,\alpha(t-))u\Big)^2\Big]\lambda_j(t)\\
&+\Big(\tilde{z}-\tilde{p}(t-)\big(A_3(t,\alpha(t-))\bar{x}(t-)+C_3(t,\alpha(t-))\bar{u}(t)\big)\Big)F(t,\alpha(t-))+\frac{1}{2}\tilde{\xi}(t)u^2, \quad (4.1)
\end{aligned}
$$

where $\tilde{p}(\cdot),\tilde{q}(\cdot),\tilde{k}(\cdot),\tilde{r}(\cdot,\cdot),\tilde{s}_j(\cdot),\tilde{P}(\cdot),\tilde{R}(\cdot),\tilde{S}_j(\cdot),\tilde{z}(\cdot)$ and $\tilde{\xi}(\cdot)$ are the adjoint processes under the optimal control $\bar{u}(\cdot)$. From (3.39)-(3.42), we know that $\tilde{\xi}(\cdot)$ satisfies

$$
\mathrm{d}\tilde{\xi}(t)=\tilde{\pi}(t)\mathrm{d}W(t),\quad \tilde{\xi}(T)=\theta e^{\theta\left(G(T,\alpha(T))\bar{x}(T)^2+\frac{1}{2}\int_0^T\bar{u}(t)^2\mathrm{d}t\right)}, \tag{4.2}
$$

and $(\tilde{p}(\cdot),\tilde{q}(\cdot),\tilde{k}(\cdot),\tilde{r}(\cdot,\cdot),\tilde{s}_j(\cdot))$ satisfies

$$
\begin{cases}
\mathrm{d}\tilde{p}(t)=-\Big[A_1(t,\alpha(t))\tilde{p}(t)+A_2(t,\alpha(t))\tilde{q}(t)+A_3(t,\alpha(t))\tilde{k}(t)\\
\qquad+\int_{\mathbb{R}_0}A_4(t,\alpha(t),\zeta)\tilde{r}(t,\zeta)\nu_\alpha(\mathrm{d}\zeta)+\sum_{j=1}^{D}A_5^j(t,\alpha(t))\tilde{s}_j(t)\lambda_j(t)\Big]\mathrm{d}t\\
\qquad+\tilde{q}(t)\mathrm{d}W(t)+\tilde{k}(t)\mathrm{d}B(t)+\int_{\mathbb{R}_0}\tilde{r}(t,\zeta)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}\zeta)+\sum_{j=1}^{D}\tilde{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t),\\
\tilde{p}(T)=2\theta e^{\theta\left(G(T,\alpha(T))\bar{x}(T)^2+\frac{1}{2}\int_0^T\bar{u}(t)^2\mathrm{d}t\right)}G(T,\alpha(T))\bar{x}(T),
\end{cases} \tag{4.3}
$$

where $\bar{x}(\cdot)$ is the solution of (2.8) under $\bar{u}(\cdot)$. Now suppose (4.2) and (4.3) admit unique solutions $(\tilde{\xi}(t),\tilde{\pi}(t))\in L^2_{\mathcal{F}}([0,T];\mathbb{R})\times L^2_{\mathcal{F}}([0,T];\mathbb{R})$ and $(\tilde{p}(t),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),(\tilde{s}_j(t))_{j=1,\dots D})\in L^2_{\mathcal{F}}([0,T];\mathbb{R})\times L^2_{\mathcal{F},p}([0,T];\mathbb{R})\times L^2_{\mathcal{F},p}([0,T];\mathbb{R})\times F^2_p([0,T];\mathbb{R})\times M^2_p([0,T];\mathbb{R}^D)$. By the comparison principle for BSDE (4.2), we deduce that $\tilde{\xi}(t)>0$ for all $0\le t\le T$, P-a.s. Then, from Theorem 3.5, if $\bar{u}(\cdot)$ is optimal, then it satisfies

$$
\begin{aligned}
\bar{u}(t)=E^{\bar{u}}\Big[-\frac{1}{\tilde{\xi}(t)}\Big[&C_1(t,\alpha(t-))\tilde{p}(t-)+C_2(t,\alpha(t-))\tilde{q}(t)+C_3(t,\alpha(t-))\tilde{k}(t)\\
&+\int_{\mathbb{R}_0}C_4(t,\alpha(t-),\zeta)\tilde{r}(t,\zeta)\nu_\alpha(\mathrm{d}\zeta)+\sum_{j=1}^{D}C_5^j(t,\alpha(t-))\tilde{s}_j(t)\lambda_j(t)\Big]\Big|\mathcal{F}_t^Y\Big].
\end{aligned} \tag{4.4}
$$

Furthermore, in view of the sufficient condition given in Theorem 3.7, we can check that $g(x,e_i)=G(t,e_i)x^2$ is obviously convex in x. Meanwhile, $Z(t)$ is $\mathcal{F}_t^Y$-adapted and $\bar{u}(\cdot)$ in (4.4) satisfies the condition (3.46). On the other hand,

$$
\begin{aligned}
&\widetilde{H}(t,x,u,\alpha(t-),\tilde{p}(t-),\tilde{q}(t),\tilde{k}(t),\tilde{r}(t,\cdot),\tilde{s}(t),\tilde{z}(t)-\tilde{p}(t-)^\top\beta(t),\tilde{\xi}(t))\\
&\quad=\Big(A_1(t,\alpha(t-))x+C_1(t,\alpha(t-))u\Big)\tilde{p}(t-)+\Big(A_2(t,\alpha(t-))x+C_2(t,\alpha(t-))u\Big)\tilde{q}(t)\\
&\qquad+\Big(A_3(t,\alpha(t-))x+C_3(t,\alpha(t-))u\Big)\tilde{k}(t)
\end{aligned}
$$

$$+\int_{\mathbb{R}_0}\Big(A_4(t,\alpha(t-),\zeta)x+C_4(t,\alpha(t-),\zeta)u\Big)\tilde{r}(t,\zeta)\nu_\alpha(\mathrm{d}\zeta)$$
$$+\sum_{j=1}^{D}\Big(A_5^j(t,\alpha(t-))x+C_5^j(t,\alpha(t-))u\Big)\tilde{s}_j(t)\lambda_j(t)$$
$$+\Big(\tilde{z}-\tilde{p}(t-)\big(A_3(t,\alpha(t-))x+C_3(t,\alpha(t-))\bar{u}(t)\big)\Big)F(t,\alpha(t-))+\frac{1}{2}\tilde{\xi}(t)u^2, \tag{4.5}$$

which is linear in x and quadratic in u. This implies the above Hamiltonian function $\widetilde{H}$ is convex in (x,u). Therefore, Theorem 3.7 shows that (4.4) is indeed optimal in this case.

We now summarize the above analysis in the following corollary.

Corollary 4.1. *The optimal control for the LQ risk-sensitive control problem is given by* (4.4)*.*

Remark 4.2. *In this example, we give the optimal control only in a conditional expectation form as* (4.4)*. Actually, it is difficult to get an explicitly observable optimal control for the partially observed risk-sensitive control problem with both Poisson random jumps and regime-switching. To our knowledge, this is still an open problem.*

5. Fully Observed Risk-Sensitive Maximum Principle

As a natural deduction of the main results in Section 3, we derive a maximum principle for a fully observed risk-sensitive optimal control problem. Then we apply the obtained maximum principle to solve a risk-sensitive portfolio optimization problem in a Markov regime-switching financial market in next section.

Write, for each $t\in[0,T]$, $\mathcal{G}_t:=\mathcal{F}_t^\alpha\vee\mathcal{F}_t^W\vee\mathcal{F}_t^N$, where $\mathcal{F}_t^\alpha$, $\mathcal{F}_t^W$ and $\mathcal{F}_t^N$ are the right-continuous, P-completed filtration generated by the Markov chain $\alpha(\cdot)$, the N-dimensional standard Brownian motion $W(\cdot)$ and the M-dimensional Poisson random measure $N(\cdot,\cdot)$, respectively. We assume throughout this section that we can fully observe the filtration $\mathcal{G}_t$ at time t. A control process $u(\cdot):[0,T]\times\Omega\to U$ is called admissible if it is $\mathcal{G}_t$-predictable and satisfies $\sup_{t\in[0,T]}E|u(t)|^m<\infty$, $m=1,2,\ldots$. Write $\mathcal{A}[0,T]$ for the set of all admissible controls.

In this setting, the correlation coefficient β between the system and the observation becomes 0 and the cost functional (2.14) subject to (2.8) and (2.10) reduces to the fully observed case, i.e. minimize the cost functional

$$J(x_0,e_i;u(\cdot)):=E\Big[e^{\theta\left(g(x(T),\alpha(T))+\int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\right)}\Big], \tag{5.1}$$

over $u(\cdot)\in\mathcal{A}[0,T]$ subject to (2.6).

Define the Hamiltonian function $\hat{H}:[0,T]\times\mathbb{R}^L\times U\times S\times\mathbb{R}^L\times\mathbb{R}^{L\times N}\times\mathcal{L}^2(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0),\nu_\alpha;\mathbb{R}^{L\times M})\times\mathcal{M}^2(\mathbb{R}^+;\mathbb{R}^{L\times D})\times\mathbb{R}\to\mathbb{R}$ as follows:

$$\begin{aligned}\hat{H}(t,x,u,e_i,\hat{p},\hat{q},\hat{r},\hat{s},\hat{\xi}):=&\langle\hat{p},b(t,x,u,e_i)\rangle+\sum_{j=1}^N\langle\hat{q}_j,\sigma^j(t,x,u,e_i)\rangle\\&+\sum_{j=1}^M\int_{\mathbb{R}_0}\langle\hat{r}_j(t,\zeta),\eta^j(t,x,u,e_i,\zeta)\rangle\nu_{e_i}^j(\mathrm{d}\zeta)\\&+\sum_{j=1}^D\langle\hat{s}_j,\gamma^j(t,x,u,e_i)\rangle\lambda_{ij}+\langle\hat{\xi},f(t,u,e_i)\rangle.\end{aligned}\tag{5.2}$$

Combining Section 3.2 and Theorem 3.7, we get the following corollary.

Corollary 5.1. *Let Assumptions* $(\mathcal{A}1)$-$(\mathcal{A}5)$ *hold and let* $(\bar{u}(\cdot),\bar{x}(\cdot))$ *be a candidate optimal pair. Suppose* $(\hat{\xi}(\cdot),\hat{\pi}(\cdot))$ *and* $(\hat{p}(\cdot),\hat{q}(\cdot),\hat{r}(\cdot,\cdot),\hat{s}(\cdot))$ *are solutions of the following BSDEs:*

$$\begin{cases}\mathrm{d}\hat{\xi}(t)=\hat{\pi}(t)\mathrm{d}W(t),\\ \hat{\xi}(T)=\theta e^{\theta\left(g(\bar{x}(T),\alpha(T))+\int_0^T f(t,\bar{u}(t),\alpha(t))\mathrm{d}t\right)},\end{cases}\tag{5.3}$$

$$\begin{cases} \mathrm{d}\hat{p}(t) = -\hat{H}_x(t,\bar{x}(t),\bar{u}(t),\alpha(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{s}(t))\mathrm{d}t \\ \qquad\qquad +\hat{q}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0}\hat{r}(t,\zeta)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}\zeta) + \hat{s}(t)\mathrm{d}\widetilde{\Phi}(t), \\ \hat{p}(T) = \theta e^{\theta\left(g(\bar{x}(T),\alpha(T)) + \int_0^T f(t,\bar{u}(t),\alpha(t))\mathrm{d}t\right)} g_x(\bar{x}(T),\alpha(T)), \end{cases} \tag{5.4}$$

such that, for any $u(\cdot)\in\mathcal{A}[0,T]$, we have

$$E\int_0^T |\hat{H}_u(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\hat{p}(t-),\hat{q}(t),\hat{r}(t,\cdot),\hat{s}(t),\hat{\xi}(t))|^2\mathrm{d}t < +\infty.$$

Moreover, suppose for all $(t,x,u,e_i)\in[0,T]\times\mathbb{R}^L\times U\times S$, the Hamiltonian $\hat{H}$ is convex in (x,u) and g is convex in x, and

$$\begin{aligned} &\hat{H}(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\hat{p}(t-),\hat{q}(t),\hat{r}(t,\cdot),\hat{s}(t),\hat{\xi}(t)) \\ &\quad = \min_{u\in U}\hat{H}(t,\bar{x}(t-),u,\alpha(t-),\hat{p}(t-),\hat{q}(t),\hat{r}(t,\cdot),\hat{s}(t),\hat{\xi}(t)), \quad \text{a.e. } t\in[0,T],\ P\text{-a.s.} \end{aligned} \tag{5.5}$$

Then $\bar{u}(\cdot)$ is an optimal control and $\bar{x}(\cdot)$ is the corresponding optimal state process.

6. Application to Risk-Sensitive Portfolio Optimization under Regime-Switching

In this section, we will apply Corollary 5.1 to solve a fully observed risk-sensitive portfolio optimization problem in a Markov regime-switching financial market.

Suppose the financial market consists of one risk-free asset and L risky assets. The risk-free asset's price process $S_0(t)$ is governed by:

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t, \qquad S_0(0) > 0,$$

where $r(t)$ denotes the risk-free interest rate at time t, for each $t\in[0,T]$. Here to simplify our analysis, we assume that $r(t)$ is a deterministic function of t.

The price processes of the other L risky assets $S_k(t), k=1,\ldots,L$, are modeled by the following Markovian regime-switching geometric Brownian motions:

$$\mathrm{d}S_k(t) = S_k(t)\Big\{\mu_k(t,\alpha(t))\mathrm{d}t + \sum_{j=1}^N \sigma_{kj}(t,\alpha(t))\mathrm{d}W_j(t)\Big\}, \qquad S_k(0)>0,$$

where $\mu_k(t,e_i)$ and $\sigma_k(t,e_i) := (\sigma_{k1}(t,e_i),\ldots,\sigma_{kN}(t,e_i))$ are bounded and deterministic functions representing respectively the appreciation rate and the volatility coefficients of the kth risky asset at time t when the economy is in state e_i at that time. Throughout this section, we assume for all $t\in[0,T]$ and $e_i\in S$ that $\mu_k(t,e_i) > r(t)$ and the non-degeneracy condition on the diffusion matrix $\Sigma(t,e_i)$ defined below is satisfied, that is,

$$\Sigma(t,e_i) := \sigma(t,e_i)\sigma(t,e_i)^\top \geq \delta\mathbf{I},$$

where δ is some positive constant and $\sigma(t,e_i) := (\sigma_1(t,e_i),\ldots,\sigma_L(t,e_i))^\top = (\sigma_{kj}(t,e_i))_{L\times N}$.

Denote by $\hat{\mathcal{G}}_t := \mathcal{F}_t^\alpha \vee \mathcal{F}_t^W$ the σ-field generated by the Markov chain and the Brownian motion up to time t. Let $u_k(t),\ k=0,1,\ldots,L$, be the amount of the wealth invested in the kth asset at time t and we call $u(\cdot) := (u_1(\cdot),\ldots,u_L(\cdot))^\top$ a portfolio of the investment. The class of admissible portfolios is the set

$$\hat{\mathcal{A}}[0,T] := \Big\{u(\cdot)|u(t):[0,T]\times\Omega\to\mathbb{R}^L \text{ is } \hat{\mathcal{G}}_t\text{-predictable and satisfies } E\int_0^T |u(t)|^2\mathrm{d}t < \infty\Big\}.$$

Given any initial wealth $x(0)=x_0\geq 0$ and an admissible portfolio $u(\cdot)$, the wealth process $x^u(\cdot)$ satisfies the following stochastic differential equation:

$$\mathrm{d}x^u(t) = \big[r(t)x^u(t) + u(t)^\top B(t,\alpha(t))\big]\mathrm{d}t + u(t)^\top\sigma(t,\alpha(t))\mathrm{d}W(t),$$

where

$$B(t,e_i) := (\mu_1(t,e_i)-r(t),\ldots,\mu_L(t,e_i)-r(t))^\top,\ i=1,\ldots,D.$$

Define $\hat{x}^u(t) := x^u(t)e^{-\int_0^t r(s)\mathrm{d}s}$. It follows that

$$\mathrm{d}\hat{x}^u(t) = u(t)^\top B(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}t + u(t)^\top\sigma(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}W(t),\ \hat{x}(0)=x_0. \tag{6.1}$$

Suppose the investor has a CARA utility function and the objective is to find an appropriate portfolio $\bar{u}(\cdot) \in \hat{\mathcal{A}}[0,T]$ such that

$$J(x_0, e_i; \bar{u}(\cdot)) = \max_{u(\cdot)\in\hat{\mathcal{A}}[0,T]} -\frac{1}{\theta}E[e^{-\theta\hat{x}^u(T)}], \tag{6.2}$$

where $\theta > 0$ is a constant representing the coefficient of absolute risk aversion. To be mathematically rigorous, we also assume that $E[e^{-2\theta\hat{x}^u(T)}] < \infty$ holds. In the above, $\bar{u}(\cdot)$ is called an optimal portfolio and $\hat{x}^{\bar{u}}(\cdot)$ is the corresponding optimal wealth process. Clearly, (6.2) subject to (6.1) is equivalent to

$$J(x_0, e_i; \bar{u}(\cdot)) = -\frac{1}{\theta} \min_{u(\cdot)\in\hat{\mathcal{A}}[0,T]} E[e^{-\theta\hat{x}^u(T)}]. \tag{6.3}$$

In this case, the Hamiltonian function (5.2) becomes:

$$\hat{H}(t,x,u,e_i,\hat{p},\hat{q},\hat{r},\hat{s},\hat{\xi}) := \langle \hat{p}, u(t)^\top B(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}\rangle + \langle \hat{q}, u(t)^\top \sigma(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}\rangle.$$

Therefore, the adjoint equation (5.4) is

$$\hat{p}(t) = -\theta e^{-\theta\hat{x}^{\bar{u}}(T)} - \int_t^T \hat{q}(s)\mathrm{d}W(s) - \sum_{j=1}^{D}\int_t^T \hat{s}_j(s)\mathrm{d}\widetilde{\Phi}_j(s). \tag{6.4}$$

Using the above hypotheses and Corollary 5.1, we deduce that

$$B(t,\alpha(t))\hat{p}(t) + \sigma(t,\alpha(t))\hat{q}(t) = 0. \tag{6.5}$$

To find a solution $(\hat{p}(\cdot), \hat{q}(\cdot), (\hat{s}_j(\cdot))_{j=1,\ldots,D})$ to (6.4), we try a process $\hat{p}(\cdot)$ of the following form:

$$\hat{p}(t) = \phi(t,\alpha(t))e^{-\theta\hat{x}^{\bar{u}}(t)}, \tag{6.6}$$

where $\phi(\cdot, e_i), i = 1,\ldots,D$, is a differentiable and deterministic function which is to be determined. It follows from (6.4) that $\phi(T, e_i) = -\theta$.

Now applying Itô's formula to (6.6) yields

$$\begin{aligned}
\mathrm{d}\hat{p}(t) =& \Big\{\phi'(t,\alpha(t)) + \frac{1}{2}\theta^2\phi(t,\alpha(t))\bar{u}(t)^\top\Sigma(t,\alpha(t))\bar{u}(t)e^{-\int_0^t 2r(s)\mathrm{d}s} \\
&- \theta\phi(t,\alpha(t))\bar{u}(t)^\top B(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s} + \sum_{j=1}^{D}\big(\phi(t,e_j)-\phi(t,\alpha(t))\big)\lambda_j(t)\Big\}e^{-\theta\hat{x}^{\bar{u}}(t)}\mathrm{d}t \\
&- \theta\phi(t,\alpha(t))\bar{u}(t)^\top\sigma(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}e^{-\theta\hat{x}^{\bar{u}}(t)}\mathrm{d}W(t) \\
&+ \sum_{j=1}^{D}\big(\phi(t,e_j)-\phi(t,\alpha(t))\big)e^{-\theta\hat{x}^{\bar{u}}(t)}\mathrm{d}\widetilde{\Phi}_j(t).
\end{aligned} \tag{6.7}$$

Comparing the coefficients of (6.4) and (6.7), we have

$$\hat{q}(t) = -\theta\phi(t,\alpha(t))\bar{u}(t)^\top\sigma(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s}e^{-\theta\hat{x}^{\bar{u}}(t)}, \tag{6.8}$$

$$\hat{s}_j(t) = \big(\phi(t,e_j)-\phi(t,\alpha(t))\big)e^{-\theta\hat{x}^{\bar{u}}(t)}, \tag{6.9}$$

$$\begin{aligned}
&\phi'(t,\alpha(t)) + \frac{1}{2}\theta^2\phi(t,\alpha(t))\bar{u}(t)^\top\Sigma(t,\alpha(t))\bar{u}(t)e^{-\int_0^t 2r(s)\mathrm{d}s} \\
&\quad - \theta\phi(t,\alpha(t))\bar{u}(t)^\top B(t,\alpha(t))e^{-\int_0^t r(s)\mathrm{d}s} + \sum_{j=1}^{D}\big(\phi(t,e_j)-\phi(t,\alpha(t))\big)\lambda_j(t) = 0.
\end{aligned} \tag{6.10}$$

Substituting (6.8) into (6.5), we get

$$\bar{u}(t) = \frac{1}{\theta}\Sigma^{-1}(t,\alpha(t))B(t,\alpha(t))e^{\int_0^t r(s)\mathrm{d}s}. \tag{6.11}$$

Substituting (6.11) into (6.10), we obtain the following Markov regime-switching ordinary differential equation satisfied by $\phi(\cdot, e_i)$

$$\phi'(t, e_i) - \frac{1}{2}B(t, e_i)^\top \Sigma^{-1}(t, e_i)B(t, e_i)\phi(t, e_i) + \sum_{j=1}^{D} \big(\phi(t, e_j) - \phi(t, e_i)\big)\lambda_{ij} = 0. \tag{6.12}$$

Here, we have used the fact that for all $t \in [0, T]$, $\hat{p}(t) \neq 0$, P-a.s., in deriving the optimal portfolio.

Actually, by applying the Feynman-Kac formula to (6.12), we obtain

$$\phi(t, e_i) = -\theta E\Big[\exp\Big\{-\frac{1}{2}\int_t^T B(s, e_i)^\top \Sigma^{-1}(s, e_i)B(s, e_i)\mathrm{d}s\Big\}\Big|\alpha(t) = e_i\Big], \tag{6.13}$$

which implies for all $t \in [0, T]$ that $\phi(t, e_i) < 0$. Therefore, combining (6.6) and (6.13) leads to $\hat{p}(t) \neq 0$, P-a.s. On the other hand, substituting the optimal portfolio $\bar{u}(\cdot)$ from (6.11) into (6.1), we easily get $E[e^{-2\theta\hat{x}^{\bar{u}}(t)}] < \infty$, $t \in [0, T]$. Furthermore, it follows from problem (6.2) and (6.4) that the optimal cost functional is given by

$$J(x_0, e_i; \bar{u}(\cdot)) = \frac{1}{\theta^2}\phi(0, e_i)e^{-\theta x_0}. \tag{6.14}$$

We now summarize the discussion above in the following theorem.

Theorem 6.1. *The optimal portfolio and the corresponding optimal cost functional for the CARA utility maximization problem* (6.2) *are given by* (6.11) *and* (6.14)*, respectively.*

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